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Proofs and Examples of Concave $R(\cdot,\cdot)$ Functions

EC.1. Proof of Statements

We use ∂ to denote the derivative operator of a single variable function, ∂_x to denote the partial derivative operator of a multi-variable function with respect to variable x, and $1_{\{\cdot\}}$ to denote the indicator function. The following lemma is used throughout our proof.

LEMMA EC.1. *Let $F_i(z,Z)$ be a continuously differentiable and jointly concave function in (z,Z) for $i=1,2$, where $z\in[\underline{z},\bar{z}]$ ($\underline{z}$ and $\bar{z}$ might be infinite) and $Z\in\mathbb{R}^n$. For $i=1,2$, let*

$$(z_i,Z_i):=\arg\max_{(z,Z)} F_i(z,Z),$$

be the optimizers of $F_i(\cdot,\cdot)$. If $z_1<z_2$, we have:

$$\partial_z F_1(z_1,Z_1)\le \partial_z F_2(z_2,Z_2).$$

Proof: $z_1<z_2$, so $\underline{z}\le z_1<z_2\le\bar{z}$. Hence, $\partial_z F_1(z_1,Z_1)\begin{cases}=0 & \text{if } z_1>\underline{z},\\ \le 0 & \text{if } z_1=\underline{z};\end{cases}$ and $\partial_z F_2(z_2,Z_2)\begin{cases}=0 & \text{if } z_2<\bar{z},\\ \ge 0 & \text{if } z_2=\bar{z}.\end{cases}$ i.e., $\partial_z F_1(z_1,Z_1)\le 0\le \partial_z F_2(z_2,Z_2)$. *Q.E.D.*

Proof of Lemma 1: Since $p(\cdot)$ and $\gamma(\cdot)$ are twice continuously differentiable, $R(\cdot,\cdot)$ is twice continuously differentiable, and jointly concave in (d_t,I_t^a) if and only if the Hessian of $R(d_t,I_t^a)$ is negative semi-definite, i.e., $\partial^2_{d_t}R(d_t,I_t^a)\le 0$, and $\partial^2_{d_t}R(d_t,I_t^a)\partial^2_{I_t^a}R(d_t,I_t)\ge(\partial_{d_t}\partial_{I_t^a}R(d_t,I_t^a))^2$, where $\partial^2_{d_t}R(d_t,I_t^a)=p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t)$, $\partial_{d_t}\partial_{I_t^a}R(d_t,I_t^a)=p'(d_t)\gamma'(I_t^a)$, and $\partial^2_{I_t^a}R(d_t,I_t^a)=(p(d_t)-b-\alpha(c+r_d))\gamma''(I_t^a)$. It is easily verified that the Hessian of $R(d_t,I_t^a)$ is negative semi-definite if and only if $(p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t))(p(d_t)-b-\alpha(c+r_d))\gamma''(I_t^a)\ge(p'(d_t)\gamma'(I_t^a))^2$. *Q.E.D.*

Proof of Lemma 2: For part (a), if $\gamma''(I_t^a)=0$, the left hand side of (3) equals to 0. Since the right hand side of (3) is greater than or equal to 0 and $(p'(d_t))^2>0$, the (3) holds only if $\gamma'(I_t^a)=0$. For the second half of **part (a)**, it suffices to show that if $\gamma'(I^0)=0$, $\gamma'(I_t^a)=0$ for any $I_t^a\le I^0$. Since $\gamma''(I_t^a)\le 0$ for all $I_t^a\le K_a$, $\gamma'(I_t^a)\ge\gamma'(I^0)=0$ for any $I_t^a\le I^0$. On the other hand, $\gamma'(I_t^a)\le 0$ for all $I_t^a\le K_a$, so $\gamma'(I_t^a)=0$ and, thus, $\gamma''(I_t^a)=0$ for all $I_t^a\le I^0$.

Part (b): By **part (a)**, for any I_t^a such that $\gamma''(I_t^a)=0$, $\gamma'(I_t^a)=0$. $(\gamma'(I_t^a))^2\le -M\gamma''(I_t^a)$ for any $0<M<+\infty$. Now we suppose $\gamma''(I_t^a)\ne 0$. Since $p(\cdot)$, $p'(\cdot)$ and $p''(\cdot)$ are continuous functions defined on a compact set $[\underline{d},\bar{d}]$ with $p'(\cdot)<0$ and $\gamma(K_a)\le\gamma(I_t^a)\le\gamma_0$, $(p''(d_t)(d_t+\gamma(I_t^a))+2p'(d_t))(p(d_t)-b-\alpha(c+r_d))/(p'(d_t))^2$ is uniformly bounded from below by a constant number, and we define this number to be $-M$. Hence, by (3), $(\gamma'(I_t^a))^2 \le -M\gamma''(I_t^a)$. *Q.E.D.*

Proof of Lemma 3:

Part (a). Observe that $\hat{p}'_\delta(\cdot) \equiv p'(\cdot)$ and $\hat{p}''_\delta(\cdot) \equiv p''(\cdot)$ for any $\delta > 0$. Thus, let

$$m := \max_{d_t \in [\underline{d}, \bar{d}], I_t^a \le K_a} \{\frac{\hat{p}''_\delta(d_t)(d_t + \gamma(I_t^a)) + 2\hat{p}'_\delta(d_t)}{(\hat{p}'_\delta(d_t))^2}\} = \max_{d_t \in [\underline{d}, \bar{d}], I_t^a \le K_a} \{\frac{p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t)}{(p'(d_t))^2}\} < 0,$$

$$k := \min_{d_t \in [\underline{d}, \bar{d}]} \{p(d_t) - b - \alpha(c + r_d)\} \ge 0,$$

and

$$\delta^* := -\frac{M}{m} - k < +\infty.$$

Therefore, for any $\delta \ge \delta^*, d_t \in [\underline{d}, \bar{d}], I_t^a \le K_a$,

$$\begin{aligned}
&\frac{(\hat{p}''_\delta(d_t)(d_t + \gamma(I_t^a)) + 2\hat{p}'_\delta(d_t))(\hat{p}_\delta(d_t) - b - \alpha(c + r_d))}{(\hat{p}'_\delta(d_t))^2}\gamma''(I_t^a) \\
=&\frac{p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t)}{(p'(d_t))^2}(p(d_t) + \delta - b - \alpha(c + r_d))\gamma''(I_t^a) \\
\ge&\frac{p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t)}{(p'(d_t))^2}(-\frac{M}{m} - k + p(d_t) - b - \alpha(c + r_d))\gamma''(I_t^a) \\
\ge&\frac{p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t)}{(p'(d_t))^2} \cdot (-\frac{M}{m})\gamma''(I_t^a) \\
\ge& - M\gamma''(I_t^a) \\
\ge&(\gamma'(I_t^a))^2,
\end{aligned}$$

where the first inequality follows from $\delta \ge \delta^*$, the second from $p(d_t) - b - \alpha(c + r_d) \ge k$, the third from the definition of m and the last from the assumption that $-M\gamma''(I_t^a) \ge (\gamma'(I_t^a))^2$ for any $I_t^a \le K_a$. Hence, by (3), for any $\delta \ge \delta^*$, $\hat{R}_\delta(\cdot, \cdot)$ is jointly concave on $d_t \in [\underline{d}, \bar{d}], I_t^a \le K_a$.

Part (b). Observe that $\hat{\gamma}'_\varsigma(\cdot) \equiv \gamma'(\cdot)$ and $\hat{\gamma}''_\varsigma(\cdot) \equiv \gamma''(\cdot)$ for any $\varsigma > 0$. Since $p''(d_t) \ne 0$, let

$$n := \max_{d_t \in [\underline{d}, \bar{d}]} \{\frac{(p(d_t) - b - \alpha(c + r_d))p''(d_t)}{(p'(d_t))^2}\} < 0,\ l := \min_{d_t \in [\underline{d}, \bar{d}], I_t^a \le K_a} \{\gamma(I_t^a) + d_t + \frac{2p'(d_t)}{p''(d_t)}\} > 0,$$

and

$$\varsigma^* := -\frac{M}{n} - l < +\infty.$$

Therefore, for any $\varsigma \geq \varsigma^*, d_t \in [\underline{d}, \bar{d}], I_t^a \leq K_a$,

$$
\begin{aligned}
&\frac{(p''(d_t)(d_t+\hat{\gamma}_\varsigma(I_t^a))+2p'(d_t))(p(d_t)-b-\alpha(c+r_d))}{(p'(d_t))^2}\hat{\gamma}_\varsigma''(I_t^a)\\
=&\frac{(p(d_t)-b-\alpha(c+r_d))(p''(d_t)(d_t+\gamma(I_t^a)+\varsigma)+2p'(d_t))}{(p'(d_t))^2}\gamma''(I_t^a)\\
=&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(\varsigma+\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)})\gamma''(I_t^a)\\
\geq&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(-\frac{M}{n}-l+\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)})\gamma''(I_t^a)\\
\geq&\frac{(p(d_t)-b-\alpha(c+r_d))p''(d_t)}{(p'(d_t))^2}(-\frac{M}{n})\gamma''(I_t^a)\\
\geq&-M\gamma''(I_t^a)\\
\geq&(\gamma'(I_t^a))^2=(\hat{\gamma}_\varsigma'(I_t^a))^2,
\end{aligned}
$$

where the first inequality follows from $\varsigma \geq \varsigma^*$, the second from $\gamma(I_t^a)+d_t+\frac{2p'(d_t)}{p''(d_t)} \geq l$, the third from the definition of n and the last from the assumption that $-M\gamma''(I_t^a) \geq (\gamma'(I_t^a))^2$ for any $I_t^a \leq K_a$. Hence, by (3), for any $\varsigma \geq \varsigma^*$, $\hat{R}_\varsigma(\cdot,\cdot)$ is jointly concave on $d_t \in [\underline{d}, \bar{d}], I_t^a \leq K_a$. *Q.E.D.*

Proof of Lemma 4: We prove parts (a) - (b) together by backward induction.

We first show, by backward induction, that if $V_{t-1}(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is concavely decreasing in both I_{t-1}^a and I_{t-1}, both $g_t(x_t^a, x_t, d_t, I_t^a) := \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\}$ and $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ are jointly concave, $g_t(\cdot,\cdot,\cdot, I_t^a)$ and $J_t(\cdot,\cdot,\cdot, I_t^a, I_t)$ are strictly concave for any fixed I_t^a and I_t, and $V_t(I_t^a, I_t) - r_d I_t^a - cI_t$ is jointly concave and decreasing in I_t^a and I_t. It is clear that $V_0(I_0^a, I_0) - r_d I_0^a - cI_0 = -r_d I_0^a - cI_0$ is jointly concave, and decreasing in I_0^a and I_0. Hence, the initial condition holds.

Assume that $V_{t-1}(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is concavely decreasing in both I_{t-1}^a and I_{t-1}. Therefore, $G_t(x,y)$ is jointly concave and decreasing in x and y. For every realization of $\epsilon_t = (\epsilon_t^a, \epsilon_t^m)$, we verify that $G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))$ is jointly concave in (x_t^a, x_t, d_t, I_t^a)as follows: let $0 \leq \lambda \leq 1$, $x_*^a := \lambda x_1^a + (1-\lambda)x_2^a$, $x_* := \lambda x_1 + (1-\lambda)x_2$, $d_* := \lambda d_1 + (1-\lambda)d_2$ and $I_*^a := \lambda I_1^a + (1-\lambda)I_2^a$, we have:

$$
\begin{aligned}
&\lambda G_t(x_1^a-(d_1+\gamma(I_1))\epsilon_t^m-\epsilon_t^a, x_1-(d_1+\gamma(I_1))\epsilon_t^m-\epsilon_t^a)\\
&+(1-\lambda)G_t(x_2^a-(d_2+\gamma(I_2))\epsilon_t^m-\epsilon_t^a, x_2-(d_2+\gamma(I_2))\epsilon_t^m-\epsilon_t^a)\\
\leq&G_t(x_*^a-(d_*+\lambda\gamma(I_1)+(1-\lambda)\gamma(I_2))\epsilon_t^m-\epsilon_t^a, x_*-(d_*+\lambda\gamma(I_1)+(1-\lambda)\gamma(I_2))\epsilon_t^m-\epsilon_t^a)\\
\leq&G_t(x_*^a-(d_*+\gamma(I_*))\epsilon_t^m-\epsilon_t^a, x_*-(d_*+\gamma(I_*))\epsilon_t^m-\epsilon_t^a),
\end{aligned}
$$

where the first inequality follows from the joint concavity of $G_t(\cdot,\cdot)$, the second from the concavity of $\gamma(\cdot)$, the monotonicity that $G_t(\cdot,\cdot)$ is decreasing in both of its arguments, and $\epsilon_t^m \geq 0$. Since concavity is preserved under expectation, $g_t(x_t^a, x_t, d_t, I_t^a) = \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))\}$

is jointly concave in (x_t^a, x_t, d_t, I_t^a). Note that $R(d_t, I_t^a)$ is jointly concave in (d_t, I_t^a), $-\theta(x_t - I_t)^-$ is jointly concave in (x_t, I_t), and $-(r_d + r_w)(x_t^a - I_t^a)^-$ is jointly concave in (x_t^a, I_t^a). Therefore, $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is jointly concave in $(x_t^a, x_t, d_t, I_t^a, I_t)$. The strict concavity of $g_t(\cdot,\cdot,\cdot,I_t^a)$ follows directly from the continuous distribution of D_t and that its support is an interval. Since $g_t(\cdot,\cdot,\cdot,I_t^a)$ is strictly concave and $R(\cdot, I_t^a)$ is concave for any fixed I_t^a, $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$ is strictly jointly concave for any fixed I_t^a and I_t.

Concavity is preserved under maximization (see, e.g., Boyd and Vandenberghe 2004), so the joint concavity of $V_t(I_t^a, I_t)$ follows immediately from the joint concavity of $J_t(\cdot,\cdot,\cdot,\cdot,\cdot)$. We now verify that $V_t(I_t^a, I_t)$ is decreasing in both I_t^a and I_t. Observe that $\gamma(I_t^a)$, $-(r_d + r_w)(x_t^a - I_t^a)^-$, and $G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t))$ are decreasing in I_t^a, and $-\theta(x_t - I_t)^-$ is decreasing in I_t. Hence, $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t^a and I_t for any fixed (x_t^a, x_t, d_t). Assume $I_1^a > I_2^a$, we have $F(I_1^a) \subset F(I_2^a)$. Hence, for any I_t,

$$\begin{aligned} V_t(I_1^a, I_t) - r_d I_1^a - cI_t &= \max_{(x_t^a, x_t, d_t) \in F(I_1^a)} J_t(x_t^a, x_t, d_t, I_1^a, I_t) \\ &\le \max_{(x_t^a, x_t, d_t) \in F(I_2^a)} J_t(x_t^a, x_t, d_t, I_2^a, I_t) = V_t(I_2^a, I_t) - r_d I_2^a - cI_t, \end{aligned}$$

where the inequality follows from the monotonicity that $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t^a, and $F(I_1^a) \subset F(I_2^a)$, thus verifying $V_t(I_t^a, I_t)$ is decreasing in I_t^a. Analogously, if $I_1 > I_2$, for any I_t^a,

$$\begin{aligned} V_t(I_t^a, I_1) - r_d I_t^a - cI_1 &= \max_{(x_t^a, x_t, d_t) \in F(I_t^a)} J_t(x_t^a, x_t, d_t, I_t^a, I_1) \\ &\le \max_{(x_t^a, x_t, d_t) \in F(I_t^a)} J_t(x_t^a, x_t, d_t, I_t^a, I_2) = V_t(I_t^a, I_2) - r_d I_t^a - cI_2, \end{aligned}$$

where the inequality follows from the monotonicity that $J_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is decreasing in I_t.

Second, we show, again by backward induction, that if $V_{t-1}(\cdot,\cdot)$ is continuously differentiable, $g_t(\cdot,\cdot,\cdot,\cdot)$ and $V_t(\cdot,\cdot)$ are continuously differentiable on the interior of their domains. For $t = 0$, $V_t(I_t^a, I_t) = 0$ is clearly continuously differentiable. The initial condition holds.

Assume $V_{t-1}(I_{t-1}^a, I_{t-1})$ is continuously differentiable,

$$\begin{aligned} g_t(x_t^a, x_t, d_t, I_t) =& \mathbb{E}\{-(b + h_a)(x_t^a - (d_t + \gamma(I_t))\epsilon_t^m - \epsilon_t^a)^+ \\ & + \alpha[V_{t-1}(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\ & - r_d(x_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) - c(x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)]\}. \end{aligned}$$

Since ϵ_t^a and ϵ_t^m are continuous, it is easy to compute the partial derivatives of $g_t(\cdot,\cdot,\cdot,\cdot)$ as follows:

$$
\begin{aligned}
\partial_{x_t^a} g_t(x_t^a,x_t,d_t,I_t^a) =& \mathbb{E}\{-(b+h_a)1_{\{x_t^a \geq (d_t+\gamma(I_t))\epsilon_t^m+\epsilon_t^a\}} \\
& + \alpha\partial_{I_{t-1}^a} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\} - \alpha r_d, \\
\partial_{x_t} g_t(x_t^a,x_t,d_t,I_t^a) =& \mathbb{E}\{\alpha\partial_{I_{t-1}} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\} - \alpha c, \\
\partial_{d_t} g_t(x_t^a,x_t,d_t,I_t^a) =& \mathbb{E}\{(b+h_a)\epsilon_t^m 1_{\{x_t^a \geq (d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a\}} \\
& - \alpha\epsilon_t^m\partial_{I_{t-1}^a} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a) \\
& - \alpha\epsilon_t^m\partial_{I_{t-1}} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\} + \alpha(r_d+c), \\
\partial_{I_t^a} g_t(x_t^a,x_t,d_t,I_t^a) =& \mathbb{E}\{(b+h_a)\gamma'(I_t^a)\epsilon_t^m 1_{\{x_t^a \geq (d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a\}} \\
& - \alpha\gamma'(I_t^a)\epsilon_t^m\partial_{I_{t-1}^a} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a) \\
& - \alpha\gamma'(I_t^a)\epsilon_t^m\partial_{I_{t-1}} V_{t-1}(x_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\} + \alpha(r_d+c)\gamma'(I_t^a)
\end{aligned}
\tag{EC.0}
$$

where the exchangeability of differentiation and expectation is easily justified using the canonical argument (see, for example, Theorem A.5.1 of Durrett 2010, the condition of which can be easily checked observing the continuity of partial derivatives of $V_{t-1}(\cdot,\cdot)$, and that the distribution of D_t is continuous.). Since at least one of ϵ_t^a and ϵ_t^m follows a continuous distribution, $\partial_{x_t^a} g_t(x_t^a,x_t,d_t,I_t^a)$, $\partial_{x_t} g_t(x_t^a,x_t,d_t,I_t^a)$, $\partial_{d_t} g_t(x_t^a,x_t,d_t,I_t^a)$ and $\partial_{I_t^a} g_t(x_t^a,x_t,d_t,I_t^a)$ are continuous. Therefore, $g_t(\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable.

Since $g_t(\cdot,\cdot,\cdot,I_t^a)$ is strictly concave and continuously differentiable, $J_t(\cdot,\cdot,\cdot,I_t^a,I_t)$ is strictly concave and continuously differentiable. Moreover, $J_t(\cdot,\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable if $x_t^a \neq I_t^a$ and $x_t \neq I_t$, i.e., it is continuously differentiable almost everywhere. By envelope theorem, $V_t(\cdot,\cdot)$ is also differentiable on the interior of the feasible set $F(I_t^a)$ for $x_t^{a*}(I_t^a,I_t) \neq I_t^a$ and $x_t^*(I_t^a,I_t) \neq I_t$. For the case $x_t^{a*}(I_t^a,I_t) = I_t^a$ or $x_t^*(I_t^a,I_t) = I_t$, we show the continuous differentiability of $V_t(\cdot,\cdot)$ in the proof of Theorem 1. This completes the induction and, hence, the proof of Lemma 4. *Q.E.D.*

Proof of Theorem 1: Parts (a) - (d) and the differentiability of $V_t(I_t^a,I_t)$. We first show parts (a) - (d) and the continuous differentiability of $V_t(I_t^a,I_t)$.

Observe that if $x_t > I_t$ (i.e., the firm orders),

$$\partial_{x_t} J_t(x_t^a,x_t,d,I_t^a,I_t) = -\psi + \partial_{x_t} g_t(x_t^a,x_t,d_t,I_t^a) < 0.$$

Hence, if $x_t^*(I_t^a,I_t) > I_t$, $x_t^{a*}(I_t^a,I_t) = x_t^*(I_t^a,I_t) > I_t \geq I_t^a$ and the optimal policy is given by Equation (7). i.e., if $x_t^a(I_t^a) > I_t$, $(x_t^{a*}(I_t^a,I_t), x_t^*(I_t^a,I_t), d_t^*(I_t^a,I_t)) = (x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a))$. This completes the proof of **part (b)**.

If $x_t < I_t$ (i.e., the firm disposes), $-\theta(x_t - I_t)^- = \theta(x_t - I_t)$. Hence, the objective function

$$\begin{aligned} J_t(x_t^a, x_t, d_t, I_t^a, I_t) = & -\theta I_t + R(d_t, I_t^a) + (\theta - \psi)x_t - (r_d + r_w)(x_t^a - I_t^a)^- + \phi x_t^a \\ & + \mathbb{E}\{G_t(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t), x_t - \delta(p(d_t), I_t^a, \epsilon_t)). \end{aligned}$$

Hence, if $x_t^*(I_t^a, I_t) < I_t$, the optimizer prescribed in Equation (9) is the optimal policy. i.e., if $\tilde{x}_t(I_t^a) < I_t$, $(x_t^{a*}(I_t^a, I_t), x_t^*(I_t^a, I_t), d_t^*(I_t^a, I_t)) = (\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a))$. **Part (c)** follows.

Next we show that $x_t^a(I_t^a) \le \tilde{x}_t(I_t^a)$. If $x_t^a(I_t^a) > \tilde{x}_t(I_t^a)$, suppose $I_t \in (\tilde{x}_t(I_t^a), x_t^a(I_t^a))$. We have that:

$$\begin{cases} J_t(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a, I_t) & > \sup_{x_t^a \le I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\}, \\ J_t(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a, I_t) & > \sup_{x_t^a \le I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\}. \end{cases} \quad \text{(EC.1)}$$

By the concavity of $J_t(\cdot, \cdot, \cdot, I_t^a, I_t)$,

$$\sup_{x_t^a \le I_t, d_t \in [\underline{d}, \bar{d}]}\{J_t(x_t^a, I_t, d_t, I_t^a, I_t)\} \ge \lambda J_t(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a, I_t) + (1-\lambda) J_t(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a, I_t),$$

where $\lambda x_t^a(I_t^a) + (1-\lambda)\tilde{x}_t(I_t^a) = I_t$. The above inequality contradicts inequality (EC.1). Hence, $x_t^a(I_t^a) \le \tilde{x}_t(I_t^a)$. **Part (d)** thus follows from part (b), part (c), $x_t^a(I_t^a) \le \tilde{x}_t(I_t^a)$, and the concavity of $J_t(\cdot, \cdot, \cdot, I_t^a, I_t)$. The second part of **part (a)** summarizes parts (b) - (d).

Since the proof of Lemma 4 already shows that $J_t(\cdot, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable, it suffices to show that $V_t(I_t^a, I_t)$ is continuously differentiable when $x_t^{a*}(I_t^a, I_t) = I_t^a$ or $x_t^*(I_t^a, I_t) = I_t$, given that $J_t(\cdot, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable. We only show that $\partial_{I_t} V_t(I_t^a, I_t)$ is continuous at the points where $x_t^*(I_t^a, I_t) = I_t$, because the continuity of $\partial_{I_t^a} V_t(I_t^a, I_t)$ at the points where $x_t^{a*}(I_t^a, I_t) = I_t^a$ follows from the same approach.

By the proof of Lemma 4, it suffices to check that the left and right partial derivatives, $\partial_{I_t} V_t(I_t^a, I_t-)$ and $\partial_{I_t} V_t(I_t^a, I_t+)$, are equal when $I_t = x_t^a(I_t^a)$ and $I_t = \tilde{x}_t(I_t^a)$. For $I_t = x_t^a(I_t^a)$, by the envelope theorem,

$$\begin{cases} \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)-) = & c \\ \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)+) = & c + \beta + \partial_{x_t^a} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) + \partial_{x_t} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a). \end{cases}$$

The first order condition with respect to x_t^a and x_t implies that

$$\beta + \partial_{x_t^a} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) + \partial_{x_t} g(x_t^a(I_t^a), x_t^a(I_t^a), d_t(I_t^a), I_t^a) = 0.$$

Therefore, $\partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)-) = \partial_{I_t} V_t(I_t^a, x_t^a(I_t^a)+)$. For $I_t = \tilde{x}_t(I_t^a)$, by the envelop theorem,

$$\begin{cases} \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)-) = & c - \theta \\ \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)+) = & c - \psi + \partial_{x_t} g(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a). \end{cases}$$

The first order condition with respect to x_t at $I_t = \tilde{x}_t(I_t^a)$ implies that

$$\partial_{x_t} g(\tilde{x}_t^a(I_t^a), \tilde{x}_t(I_t^a), \tilde{d}_t(I_t^a), I_t^a) + \theta - \psi = 0.$$

Hence, $\partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)-) = \partial_{I_t} V_t(I_t^a, \tilde{x}_t(I_t^a)+)$ and the partial derivative $\partial_{I_t} V_t(I_t^a, I_t)$ is continuous.

Part (e): Let

$$J_t^a(x_t^a, d_t, I_t^a) := R(d_t, I_t^a) + \beta x_t^a + g_t^a(x_t^a, d_t, I_t^a),$$

where $g_t^a(x_t^a, d_t, I_t^a) = \mathbb{E}[G_t^a(x_t^a - \delta(p(d_t), I_t^a, \epsilon_t))]$, with $G_t^a(x) = G_t^a(x, x)$.

We first show that $x_t^a(I_t^a)$ is decreasing in I_t^a. Let $\gamma_t := \gamma(I_t^a)$ and $y_t := d_t + \gamma_t$. Then, we have $J_t^a(x_t^a, d_t, I_t^a) = \hat{J}_t^a(x_t^a, y_t, \gamma_t)$, where

$$\hat{J}_t^a(x_t^a, y_t, \gamma_t) = R^*(y_t, \gamma_t) + \beta x_t^a + \mathbb{E}\{G_t^a(x_t^a - y_t\epsilon_t^m - \epsilon_t^a)\},$$

with $R^*(y_t, \gamma_t) := R(y_t - \gamma_t, I_t^a)$. We need the following lemma that establishes the supermodularity of $R^*(\cdot, \cdot)$ and $R(\cdot, \cdot)$:

Lemma EC.2. *(a) $R^*(y_t, \gamma_t)$ is strictly supermodular in (y_t, γ_t), where $y_t - \gamma_t = d_t \in [\underline{d}, \bar{d}]$ and $y_t \geq 0$. In addition, $R^*(y_t, \gamma_t)$ is strictly concave in y_t, for any fixed γ_t;*

(b) $R(d_t, I_t^a)$ is supermodular in (d_t, I_t^a), where $d_t \in [\underline{d}, \bar{d}]$ and $I_t^a \leq K_a$. In addition, $R(d_t, I_t^a)$ is strictly concave in d_t, for any fixed I_t^a.

Proof of Lemma EC.2: $R^*(y_t, \gamma_t) = (p(y_t - \gamma_t) - b - \alpha(c + r_d))y_t$ is twice continuously differentiable when $y_t - \gamma_t = d_t \in [\underline{d}, \bar{d}]$ and $y_t \geq 0$. To prove the supermodularity of $R^*(\cdot, \cdot)$, it suffices to show that $\partial_{y_t}\partial_{\gamma_t} R^*(y_t, \gamma_t) \geq 0$. Direct computation yields that: $\partial_{y_t}\partial_{\gamma_t} R^*(y_t, \gamma_t) = -(p''(y_t - \gamma_t)y_t + p'(y_t - \gamma_t))$. Since $p'(\cdot) < 0$ and $p''(\cdot) \leq 0$, $-(p''(y_t - \gamma_t)y_t + p'(y_t - \gamma_t)) > 0$. Hence, $R^*(y_t, \gamma_t)$ is strictly supermodular. Moreover, $\partial_{y_t}^2 R^*(y_t, \gamma_t) = p''(y_t - \gamma_t)y_t + 2p'(y_t - \gamma_t) < 0$, since $p''(\cdot) \leq 0$ and $p'(\cdot) < 0$. Hence, $R^*(y_t, \gamma_t)$ is strictly concave in y_t, for any fixed γ_t. This establishes part (a).

$R(\cdot, \cdot)$ is twice continuously differentiable, $\partial_{d_t}\partial_{I_t^a} R(d_t, I_t^a) = p'(d_t)\gamma'(I_t^a) \geq 0$. Hence, $R(\cdot, \cdot)$ is supermodular. In addition, $\partial_{d_t}^2 R(d_t, I_t^a) = p''(d_t)(d_t + \gamma(I_t^a)) + 2p'(d_t) < 0$, so $R(d_t, I_t^a)$ is strictly concave in d_t for any fixed I_t^a. *Q.E.D.*

As shown in the proof of Lemma 4, $G_t(\cdot, \cdot)$ and, thus, $G_t^a(\cdot)$, is concave. Note that $\epsilon_t^m \geq 0$, so, for any realization of $(\epsilon_t^a, \epsilon_t^m)$, it is easily verified that $G_t^a(x_t - y_t\epsilon_t^m - \epsilon_t^a)$ is supermodular in (x_t, y_t). Hence, $\mathbb{E}\{G_t^a(x_t - y_t\epsilon_t^m - \epsilon_t^a)\}$ is supermodular in (x_t, y_t), since supermodularity is preserved under expectation. By Lemma EC.2, $R^*(y_t, \gamma_t)$ is supermodular and, thus, $\hat{J}_t^a(x_t, y_t, \gamma_t)$ is supermodular in (x_t, y_t, γ_t). Therefore, the optimal order-up-to level, $x_t^a(I_t^a)$, and optimal expected demand $y_t(I_t^a) := d_t(I_t^a) + \gamma_t$ are increasing in γ_t, and, since $\gamma(\cdot)$ is decreasing in I_t^a, decreasing in I_t^a.

We now proceed to show that the optimal expected price-induced demand $d_t(I_t^a)$ is increasing in I_t^a. Let $I_1^a > I_2^a$, $x_1^a := x_t^a(I_1^a)$, $x_2^a := x_t^a(I_2^a)$, $d_1 := d_t(I_1^a)$, $d_2 := d_t(I_2^a)$ $y_1 := d_1 + \gamma(I_1^a)$, and $y_2 :=$

$d_2+\gamma(I_2^a)$. We prove that $d_1 \geq d_2$ by contradiction. Assume that $d_1 < d_2$. By Lemma EC.1, $d_1 < d_2$ implies that $\partial_{d_t} J_t^a(x_1^a, d_1, I_1^a) \leq \partial_{d_t} J_t^a(x_2^a, d_2, I_2^a)$.

$$\partial_{d_t} R(d_1, I_1^a) \geq \partial_{d_t} R(d_1, I_2^a) > \partial_{d_t} R(d_2, I_2^a),$$

where the first inequality follows from the supermodularity of $R(\cdot,\cdot)$ and the second inequality follows from the strict concavity of $R(\cdot, I_t^a)$. Hence,

$$\partial_{d_t} g_t^a(x_1^a, d_1, I_1^a) = \partial_{d_t} J_t^a(x_1^a, d_1, I_1^a) - \partial_{d_t} R(d_1, I_1^a) < \partial_{d_t} J_t^a(x_2^a, d_2, I_2^a) - \partial_{d_t} R(d_2, I_2^a) = \partial_{d_t} g_t^a(x_2^a, d_2, I_2^a). \tag{EC.2}$$

Let

$$f(X) := -(b+h_a)1_{\{X\geq 0\}} + \alpha[\partial_{I_{t-1}^a} V_{t-1}(X, X) + \partial_{I_{t-1}} V_{t-1}^a(X, X) - r_d - c] \leq 0,$$

which is decreasing in X. We have:

$$\partial_{x_t^a} g_t^a(x_i^a, d_i, I_i^a) = \mathbb{E}\{f(x_i^a - y_i\epsilon_t^m - \epsilon_t^a)\} \text{ and } \partial_{d_t} g_t^a(x_i^a, d_i, I_i^a) = \mathbb{E}\{-\epsilon_t^m f(x_i^a - y_i\epsilon_t^m - \epsilon_t^a)\} \text{ for } i = 1, 2.$$

Recall that we have proved $x_2^a \geq x_1^a$ and $y_2^a \geq y_1^a$.

If $x_1^a = x_2^a$, $x_1^a - y_1^a\epsilon_t^m - \epsilon_t^a \geq x_2^a - y_2^a\epsilon_t^m - \epsilon_t^a$ for any realization of $(\epsilon_t^a, \epsilon_t^m)$. Hence,

$$\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) = \mathbb{E}\{f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a)\} \leq \mathbb{E}\{f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)\} = \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a),$$

where the inequality follows from that $f(\cdot)$ is decreasing.

If $x_2^a > x_1^a$, by Lemma EC.1, $\partial_{x_t^a} J_t^a(x_1^a, d_1, I_1^a) \leq \partial_{x_t^a} J_t^a(x_2^a, d_2, I_2^a)$ and, hence,

$$\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) = \partial_{x_t^a} J_t^a(x_1^a, d_1, I_1^a) - \beta \leq \partial_{x_t^a} J_t^a(x_2^a, d_2, I_2^a) - \beta = \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a).$$

Note that there exists an ϵ_t^*, such that $x_1^a - y_1\epsilon_t^m \leq x_2^a - y_2\epsilon_t^m$ if $\epsilon_t^m \leq \epsilon_t^*$ and $x_1^a - y_1^a\epsilon_t^m > x_2^a - y_2^a\epsilon_t^m$ if $\epsilon_t^m > \epsilon_t^*$ (ϵ_t^* may equal $\underline{m}$ or $\overline{m}$.). Since $f(\cdot)$ is decreasing, $f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a) \geq 0$ for any $\epsilon_t^m \in [\underline{m}, \epsilon_t^*]$ and any realization of ϵ_t^a. So

$$-\epsilon_t^m(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)) \geq -\epsilon_t^*(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a)), \tag{EC.3}$$

for any $\epsilon_t^m \in [\underline{m}, \epsilon_t^*]$ and any realization of ϵ_t^a. Analogously, for $\epsilon_t^m \in [\epsilon_t^*, \overline{m}]$, $f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a) \leq 0$, and (EC.3) holds for $\epsilon_t^m \in [\epsilon_t^*, \overline{m}]$ as well. Therefore, (EC.3) holds for all $\epsilon_t^m \in [\underline{m}, \overline{m}]$ and any realization of ϵ_t^a.

Taking expectation, we have:

$$\begin{aligned}\partial_{d_t} g_t^a(x_1^a, d_1, I_1^a) - \partial_{d_t} g_t^a(x_2^a, d_2, I_2^a) &= \mathbb{E}\{-\epsilon_t^m(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a))\}\\ &\geq \mathbb{E}\{-\epsilon_t^*(f(x_1^a - y_1\epsilon_t^m - \epsilon_t^a) - f(x_2^a - y_2\epsilon_t^m - \epsilon_t^a))\}\\ &= -\epsilon_t^*(\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) - \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a))\\ &\geq 0,\end{aligned} \tag{EC.4}$$

where the last inequality follows from $\partial_{x_t^a} g_t^a(x_1^a, d_1, I_1^a) \le \partial_{x_t^a} g_t^a(x_2^a, d_2, I_2^a)$. (EC.4) contradicts (EC.2) and, hence, $d_1 \ge d_2$, i.e., $d_t(I_t^a)$ is increasing in I_t^a. The continuity of $x_t^a(I_t^a)$ and $d_t(I_t^a)$ follows directly from that the objective function $J_t^a(\cdot,\cdot,I_t^a)$ is strictly concave for any given I_t^a. The proof of **part (e)** follows. *Q.E.D.*

Remark EC.1. The supermodularity of $R^*(y_t, \gamma_t)$ implies that to better take advantage of the high demand induced by low inventory level, the firm should adjust its price to a level such that the expected demand will increase.

Proof of Theorem 2: If $h_w \ge \alpha c - s$, $\theta - \psi = c - s - h_w - (1-\alpha)c = \alpha c - s - h_w \le 0$. Since $g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is also decreasing in x_t, Equation (9) implies that $\tilde{x}_t(I_t^a) = \tilde{x}_t^a(I_t^a)$, for any t and I_t^a, which proves **part (a)**.

Observe that for any $(x_t^a, x_t, d_t, I_t^a, I_t)$,

$$\partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a, I_t) \ge -(\sum_{j=1}^{t} \alpha^j) h_w \ge -(\sum_{j=1}^{T} \alpha^j) h_w,\ t = T, T-1, \cdots 1,$$

where the first inequality holds as an equality if $x_j^*(I_j^a, I_j) = I_j$, for all $j \le t-1$. Hence, $\partial_{x_t} g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is uniformly bounded from below by $-(\sum_{j=1}^T \alpha^j) h_w$, for any t. Thus, if $\theta - \psi = \alpha c - h_w - s \ge (\sum_{j=1}^T \alpha^j) h_w$, $\tilde{x}_t(I_t^a) = +\infty$ for any t and I_t^a. Hence, $s_* = \alpha c - (\sum_{j=0}^T \alpha^j) h_w$. This proves **part (b)**.

If $\inf_{I_t^a < K_a} \gamma'(I_t^a) \ge -M$, for any $(x_t^a, x_t, d_t, I_t^a, I_t)$,

$$\partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a, I_t) \ge -M(\sum_{j=1}^{t} \alpha^j)(\bar{p} + h_a) \ge -M(\sum_{j=1}^{T} \alpha^j)(\bar{p} + h_a),\ t = T, T-1, \cdots 1,$$

where $\bar{p}$ is the maximum marginal revenue and h_a is the maximum marginal holding cost. Hence, $\partial_{x_t^a} g_t(x_t^a, x_t, d_t, I_t^a, I_t)$ is bounded from below by $-M(\sum_{j=1}^T \alpha^j)(\bar{p} + h_a)$, for any t. Thus, if $r_d + r_w + \phi \ge M(\sum_{j=1}^T \alpha^j)(\bar{p} + h_a)$, $\tilde{x}_t^a(I_t^a) \ge I_t^a$, for any $I_t^a \le K_a$.

If $\inf_{I_t^a < K_a} \gamma'(I_t^a) = -\infty$, $\lim_{I_t^a \to K_a} \gamma'(I_t^a) = -\infty$. Hence, for any x_t, d_t, and I_t,

$$\lim_{I_t^a \to K_a} \partial_{x_t^a} g_t(I_t^a, x_t, d_t, I_t^a, I_t) \le \alpha(\underline{p} - b - (1-\alpha)(c + r_d)) \lim_{I_t^a \to K_a} \gamma'(I_t^a) = -\infty.$$

Hence, for any r_w, and any x_t, d_t and I_t,

$$\partial_{x_t^a} J_t(I_t^a -, x_t, d_t, I_t^a, I_t) = r_d + r_w + \phi + \partial_{x_t^a} g_t(I_t^a, x_t, d_t, I_t^a, I_t) \to -\infty,\ \text{as } I_t^a \to K_a.$$

The above limit completes the proof of **Part (c)**.

For notational simplicity, we denote $x^{a*} := x_t^{a*}(I_{t-1}^a, I_{t-1})$, $x^* := x_t^*(I_{t-1}^a, I_{t-1})$ and $d^* := d_t^*(I_{t-1}^a, I_{t-1})$. Observe that

$$\partial_{I_{t-1}^a} V_{t-1}(I_{t-1}^a, I_{t-1}) \leq (\underline{p} - b - \alpha(c + r_d))\gamma'(I_{t-1}^a) + \partial_{I_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a). \tag{EC.5}$$

By Equation (EC.0),

$$\begin{cases} \partial_{x_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & \mathbb{E}\{f_1(\epsilon_{t-1}^m)\}, \\ \partial_{x_{t-1}} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & \mathbb{E}\{f_2(\epsilon_{t-1}^m)\}, \\ \partial_{I_{t-1}^a} g_{t-1}(x^{a*}, x^*, d^*, I_{t-1}^a) = & -\gamma'(I_{t-1}^a)\mathbb{E}\{\epsilon_{t-1}^m[f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)]\}, \end{cases}$$

where

$$\begin{cases} f_1(\epsilon_{t-1}^m) = & \mathbb{E}_{\epsilon_{t-1}^a}\{-(b+h_a)1_{\{x^{a*} \geq (d_t + \gamma(I_t))\epsilon_{t-1}^m + \epsilon_{t-1}^a\}} \\ & +\alpha\partial_{I_{t-2}^a} V_{t-2}(x^{a*} - (d^* + \gamma(I_{t-1}^a))\epsilon_{t-1}^m - \epsilon_{t-1}^a, x^* - (d^* + \gamma(I_{t-1}^a))\epsilon_{t-1}^m - \epsilon_{t-1}^a)\} - \alpha r_d \\ f_2(\epsilon_{t-1}^m) = & \mathbb{E}_{\epsilon_{t-1}^a}\{\alpha\partial_{I_{t-2}} V_{t-2}(x^{a*} - (d^* + \gamma(I_{t-1}^a))\epsilon_{t-1}^m - \epsilon_{t-1}^a, x^* - (d^* + \gamma(I_{t-1}^a))\epsilon_{t-1}^m - \epsilon_{t-1}^a)\} - \alpha c. \end{cases}$$

The first order conditions with respect to x_{t-1}^a and x_{t-1} suggest that

$$\mathbb{E}\{f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)\} \leq -(\phi - \psi) = -\beta.$$

Since $f_1(\cdot) \leq 0$ and $f(\cdot) \leq 0$, we have:

$$\mathbb{E}\{\epsilon_{t-1}^m[f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)]\} \leq \mathbb{E}\{\underline{m}[f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)]\} = \underline{m}\mathbb{E}\{f_1(\epsilon_{t-1}^m) + f_2(\epsilon_{t-1}^m)\} \leq -\underline{m}\beta.$$

Therefore, by inequality (EC.5),

$$\partial_{I_{t-1}^a} V_{t-1}(I_{t-1}^a, I_t) \leq (\underline{p} - b - \alpha(c + r_d) + \underline{m}\beta)\gamma'(I_{t-1}^a). \tag{EC.6}$$

So for any $d_t \in [\underline{d}, \bar{d}]$ and any x_t,

$$\begin{aligned} \partial_{x_t^a} g_t(0, x_t, d_t, I_t^a) \leq & \alpha\mathbb{E}[\partial_{I_{t-1}^a} V_{t-1}(-(d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)] \\ \leq & \alpha\mathbb{E}[(\underline{p} - b - \alpha(c + r_d) + \underline{m}\beta)\gamma'(-(d_t + \gamma(I_t))\epsilon_t^m - \epsilon_t^a)] \\ \leq & \alpha(\underline{p} - b - \alpha(c + r_d) + \underline{m}\beta)(1 - \iota)\gamma'(-\bar{D}) \\ \leq & -(r_d + r_w + \phi), \end{aligned} \tag{EC.7}$$

where the first inequality follows from equation (EC.0), the second from (EC.6), and the last from the assumption that $\alpha(\underline{p} - b - \alpha(c + r_d) + \underline{m}\beta)(1 - \iota)\gamma'(-\bar{D}) + (r_d + r_w + \phi) \leq 0$. The third inequality of (EC.7) follows from the following inequality:

$$\mathbb{E}[\gamma'(-D_t)] = \mathbb{E}_{D_t \geq \bar{D}}[\gamma'(-D_t)] + \mathbb{E}_{D_t \leq \bar{D}}[\gamma'(-D_t)] \leq 0 + \mathbb{E}_{D_t \leq \bar{D}}[\gamma'(-\bar{D})] \leq (1 - \iota)\gamma'(-\bar{D}),$$

where the first inequality follows from the concavity of $\gamma(\cdot)$ and the second inequality follows from the definition of $\bar{D}$. (EC.7) implies that $x_t^{a*}(I_t^a, I_t) = 0$ for all $I_t^a \leq K_a$ and all I_t, which completes the proof of **part (d)**. *Q.E.D.*

Before we proceed to prove the results in Section 5, we remark that $R^s_t(d_t, I^a_t)$ shares the same properties as $R(d_t, I^a_t)$. i.e., we have the following counterpart of Lemma EC.2 in the model without inventory withholding:

LEMMA EC.3. *(a) $R^{s*}(y_t,\gamma_t)$ is strictly supermodular in (y_t,γ_t), where $R^{s*}(y_t,\gamma_t) := R^s(y_t - \gamma_t, I^a_t)$, $y_t - \gamma_t = d_t \in [\underline{d}, \bar{d}]$ and $y_t \geq 0$. In addition, $R^{s*}(y_t,\gamma_t)$ is strictly concave in y_t, for any fixed γ_t;*

(b) $R^s(d_t, I^a_t)$ is supermodular in (d_t, I^a_t), where $d_t \in [\underline{d}, \bar{d}]$ and $I^a_t \leq K_a$. In addition, $R^s(d_t, I^a_t)$ is strictly concave in d_t, for any fixed I^a_t.

Proof of Lemma EC.3: The proof is identical to that of Lemma EC.2, and hence omitted. *Q.E.D.*

Proof of Theorem 3: The proof is very similar to that of Lemma 4 and Theorem 1, so we only sketch it.

For **parts (a) - (c)**, the proof is exactly the same as that of Lemma 4, and hence omitted.

To show **parts (d) - (f)**, we define the following unconstrained optimizers:

$$(x^L_t(I^a_t), d^L_t(I^a_t)) := \underset{x^a_t \leq K_a, d_t \in [\underline{d}, \bar{d}]}{\arg\max} \{R^s(d_t, I^a_t) + \beta^s x^a_t + \mathbb{E}[G^s_t(x^a_t - \delta(p(d_t), I^a_t, \epsilon_t))]\},$$

and

$$(x^H_t(I^a_t), d^H_t(I^a_t)) := \underset{x^a_t \leq K_a, d_t \in [\underline{d}, \bar{d}]}{\arg\max} \{R^s(d_t, I^a_t) + (\beta^s + \theta) x^a_t + \mathbb{E}[G^s_t(x^a_t - \delta(p(d_t), I^a_t, \epsilon_t))]\}.$$

We need the following lemma:

LEMMA EC.4. *Let $\gamma_t := \gamma(I^a_t)$, $\Psi(x^a_t, y_t, \mu|\gamma_t) := R^{s*}(y_t,\gamma_t) + \mu x^a_t + \mathbb{E}\{G^s_t(x^a_t - y_t\epsilon^m_t - \epsilon^a_t)\}$ is supermodular in (x^a_t, y_t, μ) for any given γ_t.*

Proof of Lemma EC.4: Since $G^s_t(\cdot)$ is concave and $\epsilon^m_t \geq 0$, $\mathbb{E}\{G^s_t(x^a_t - y_t\epsilon^m_t - \epsilon^a_t)\}$ is supermodular in (x^a_t, y_t). It's also clear that μx^a_t is strictly supermodular in (x^a_t, μ). Therefore, $\Psi(x^a_t, y_t, \mu|\gamma_t)$ is supermodular in (x^a_t, y_t, μ) for any given γ_t. *Q.E.D.*

Lemma EC.4 and its proof imply that $x^L_t(I^a_t) < x^H_t(I^a_t)$ since $\beta^s + \theta > \beta^s$. Exactly the same argument as in the proof of Theorem 1(e) implies that $x^L_t(I^a_t)$ and $x^H_t(I^a_t)$ are continuously decreasing in I^a_t and $d^L_t(I^a_t)$ and $d^H_t(I^a_t)$ are continuously increasing in I^a_t. $I^L_t := \sup\{I^a_t : I^a_t < x^L_t(I^a_t)\}$ and $I^H_t := \inf\{I^a_t : I^a_t > x^H_t(I^a_t)\}$. It's clear that I^L_t and I^H_t are the thresholds in **part (d)**. Therefore,

$$x^{s*}_t(I^a_t) = \begin{cases} x^L_t(I^a_t) & \text{if } I^a_t < I^L_t; \\ I^a_t & \text{if } I^L_t \leq I^a_t \leq I^H_t; \\ x^H_t(I^a_t) & \text{if } I^a_t > I^H_t. \end{cases}$$

It's clear that $x_t^{s*}(I_t^a)$ satisfies the statement in **part (e)**. Therefore, we have

$$d_t^{s*}(I_t^a) = \begin{cases} d_t^L(I_t^a) & \text{if } I_t^a < I_t^L; \\ \arg\max_{d_t \in [\underline{d}, \bar{d}]} J_t^s(I_t^a, d_t, I_t^a) & \text{if } I_t^L \le I_t^a \le I_t^H; \\ d_t^H(I_t^a) & \text{otherwise.} \end{cases}$$

To prove **part (f)**, it remains to show that $d_t^{s*}(I_t^a)$ is increasing in I_t^a for $I_t^L \le I_t^a \le I_t^H$. Let $U_t^s(d_t, I_t^a) := J_t^s(I_t^a, d_t, I_t^a)$ and it is easily verified that $U_t^s(d_t, I_t^a)$ is supermodular in (d_t, I_t^a). Thus, $d_t^{s*}(I_t^a)$ is increasing in I_t^a, which completes the proof of Theorem 3. *Q.E.D.*

Proof of Theorem 4: We show both parts by backward induction.

For **part (a)**, we use backward induction to recursively show this result. For $t=0$, $V_0^s(\cdot) = \hat{V}_0^s(\cdot) = 0$ and, hence, $\partial_{I_0^a} V_0^s(I_0^a) = \partial_{I_0^a} \hat{V}_0^s(I_0^a)$ for all I_0^a. We show that: if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, (a) $I_t^L \le \hat{I}_t^L$, (b) $I_t^H \le \hat{I}_t^H$, (c) $x_t^{s*}(I_t^a) \le \hat{x}_t^{s*}(I_t^a)$, (d) $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$ and (e) $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \le K_a$. To prove these inequalities, we define $(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a))$ and $(\hat{x}_t^H(I_t^a), \hat{d}_t^H(I_t^a))$ as the unconstrained optimizers in the model with demand $\hat{D}_t$, corresponding to $(x_t^L(I_t^a), d_t^L(I_t^a))$ and $(x_t^H(I_t^a), d_t^H(I_t^a))$, respectively. Let $y_t^L(I_t^a) := d_t^L(I_t^a) + \gamma(I_t^a)$, $\hat{y}_t^L(I_t^a) := \hat{d}_t^L(I_t^a) + \hat{\gamma}(I_t^a) = \hat{d}_t^L(I_t^a) + \gamma_0$, $\hat{R}^s(d_t, I_t^a) := R^s(d_t, -\infty)$, and $\hat{G}_t^s(y) := -(h_a+b)y^+ + \alpha[\hat{V}_{t-1}^s(y) - cy]$. We define the objective functions $J_t^L(x_t^a, d_t, I_t^a) := R^s(d_t, I_t^a) + \beta^s x_t^a + g_t^s(x_t^a, d_t, I_t^a)$, $\hat{J}_t^L(x_t^a, d_t, I_t^a) := \hat{R}^s(d_t, I_t^a) + \beta^s x_t^a + \hat{g}_t^s(x_t^a, d_t, I_t^a)$, where $\hat{g}_t^s(x_t^a, d_t, I_t^a) := \mathbb{E}\{\hat{G}_t^s(x_t^a - \hat{\delta}(p(d_t), I_t^a, \epsilon_t))\}$. Since $\epsilon_t^m = 1$ with probability 1, $g_t^s(x_t^a, d_t, I_t^a) = H_t^s(x_t^a - d_t - \gamma(I_t^a))$ and $\hat{g}_t^s(x_t^a, d_t, I_t^a) = \hat{H}_t^s(x_t^a - d_t - \gamma_0)$, where $H_t^s(X) := \mathbb{E}\{G_t^s(X - \epsilon_t^a)\}$ and $\hat{H}_t^s(X) := \mathbb{E}\{\hat{G}_t^s(X - \epsilon_t^a)\}$.

First, we show that, if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$, $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, $x_t^H(I_t^a) \le \hat{x}_t^H(I_t^a)$, and $d_t^H(I_t^a) \ge \hat{d}_t^H(I_t^a)$. Since $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$, $\partial_X H_t^s(X) \le \partial_X \hat{H}_t^s(X)$ for any X. We only show that $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$ and $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, while $x_t^H(I_t^a) \le \hat{x}_t^H(I_t^a)$ and $d_t^H(I_t^a) \ge \hat{d}_t^H(I_t^a)$ follow from the same argument.

We show by contradiction that $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$ and $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$. Note that, for the model with inventory-independent demand (i.e., the firm faces $\hat{D}_t$), it is reduced to the classical joint pricing and inventory management problem with stochastic demand introduced in Federgruen and Heching (1999). Hence, $\hat{x}_t^L(I_t^a)$ and $\hat{d}_t^L(I_t^a)$ are constants independent of I_t^a.

Assume that $x_t^L(I_t^a) > \hat{x}_t^L(I_t^a)$. Lemma EC.1 yields that $\partial_{x_t^a} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) \ge \partial_{x_t^a} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a)$. Hence,

$$\begin{aligned} \partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) &= \partial_{x_t^a} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) - \beta^s \\ &\ge \partial_{x_t^a} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a) - \beta^s \\ &= \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)). \end{aligned}$$

Since $\partial_X H_t^s(X) \le \partial_X \hat{H}_t^s(X)$ for any X and both of them are strictly decreasing, $y_t^L(I_t^a) > \hat{y}_t^L(I_t^a)$. Thus, $d_t^L(I_t^a) = y_t^L(I_t^a) - \gamma(I_t^a) > \hat{y}_t^L(I_t^a) - \gamma_0 = \hat{d}_t^L(I_t^a)$. Invoking Lemma EC.1, we have $\partial_{d_t} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) \ge \partial_{d_t} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a)$, and

$$\begin{aligned}\partial_{d_t} R^s(d_t^L(I_t^a), I_t^a) =& \partial_{d_t} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) + \partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a))\\ \ge& \partial_{d_t} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a) + \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a))\\ =& \partial_{d_t} \hat{R}^s(\hat{d}_t^L(I_t^a), I_t^a)\end{aligned}$$

Since $\partial_{d_t} R^s(d_t, I_t^a) = \partial_{y_t} R^{s*}(d_t + \gamma(I_t^a), \gamma(I_t^a))$, $\partial_{y_t} R^{s*}(y_t^L(I_t^a), \gamma(I_t^a)) \ge \partial_{y_t} R^{s*}(\hat{y}_t^L(I_t^a), \gamma_0)$. However, the strict concavity of $R^{s*}(\cdot, \gamma_t)$ and the supermodularity of $R^{s*}(\cdot,\cdot)$ yield that

$$\partial_{y_t} R^{s*}(y_t^L(I_t^a), \gamma(I_t^a)) < \partial_{y_t} R^{s*}(\hat{y}_t^L(I_t^a), \gamma(I_t^a)) \le \partial_{y_t} R^{s*}(\hat{y}_t^L(I_t^a), \gamma_0),$$

which leads to a contradiction. Therefore, we have $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$.

Assume that $d_t^L(I_t^a) < \hat{d}_t^L(I_t^a)$, so $y_t^L(I_t^a) = d_t^L(I_t^a) + \gamma(I_t^a) < \hat{d}_t^L(I_t^a) + \gamma_0 = \hat{y}_t^L(I_t^a)$. Lemma EC.1 yields that $\partial_{d_t} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a) \le \partial_{d_t} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a)$. The strict concavity of $R^s(\cdot, I_t^a)$ and the supermodularity of $R^s(\cdot,\cdot)$ imply that

$$\partial_{d_t} R^s(d_t^L(I_t^a), I_t^a) > \partial_{d_t} R^s(\hat{d}_t^L(I_t^a), I_t^a) \ge \partial_{d_t} R^s(\hat{d}_t^L(I_t^a), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}_t^L(I_t^a), I_t^a).$$

Hence, we have:

$$\begin{aligned}\partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) =& \partial_{d_t} R^s(d_t^L(I_t^a), I_t^a) - \partial_{d_t} J_t^L(x_t^L(I_t^a), d_t^L(I_t^a), I_t^a)\\ >& \partial_{d_t} \hat{R}^s(\hat{d}_t^L(I_t^a), I_t^a) - \partial_{d_t} \hat{J}_t^L(\hat{x}_t^L(I_t^a), \hat{d}_t^L(I_t^a), I_t^a)\\ =& \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)).\end{aligned}$$

The first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^L(I_t^a) - y_t^L(I_t^a)) = \partial_X \hat{H}_t^s(\hat{x}_t^L(I_t^a) - \hat{y}_t^L(I_t^a)) = -\beta^s$, which leads to a contradiction. Hence, $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$. We have thus proved that, if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$, $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, $x_t^H(I_t^a) \le \hat{x}_t^H(I_t^a)$, and $d_t^H(I_t^a) \ge \hat{d}_t^H(I_t^a)$. $I_t^L \le \hat{I}_t^L$ and $I_t^H \le \hat{I}_t^H$ follow immediately from $x_t^L(I_t^a) \le \hat{x}_t^L(I_t^a)$ and $x_t^H(I_t^a) \le \hat{x}_t^H(I_t^a)$.

Next, we show that $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$, for all $I_t^a \le K_a$. Since $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, $d_t^{s*}(I_t^a) = d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, for all $I_t^a \le I_t^L$. If $I_t^a \in [I_t^L, \hat{I}_t^L]$,

$$d_t^{s*}(I_t^a) \ge d_t^{s*}(I_t^L) = d_t^L(I_t^L) \ge \hat{d}_t^L(I_t^L) = \hat{d}_t^L(I_t^a),$$

where the first inequality follows from Theorem 3, the second from $d_t^L(I_t^a) \ge \hat{d}_t^L(I_t^a)$, and the last equality from Federgruen and Heching (1999) Theorem 1. If $I_t^a \in [\hat{I}_t^L, I_t^H]$ (it might be an empty set), $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a) = I_t^a$. The supermodularity of $R^s(d_t, I_t^a)$ implies that

$$\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) \ge \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a).$$

Since $\gamma_0 \ge \gamma(I_t^a)$, both $H_t^s(\cdot)$ and $\hat{H}_t^s(\cdot)$ are concave, and $\partial_X H_t^s(X) \le \partial_X \hat{H}_t^s(X)$ for all X, so $-\partial_X H_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma(I_t^a)) \ge -\partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0)$. Hence,

$$\begin{aligned}\partial_{d_t} J_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a) &= \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma(I_t^a)) \\ &\ge \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0) \\ &= \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a),\end{aligned}$$

i.e., $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$. If $I_t^a \in [I_t^H, \hat{I}_t^H]$, $x_t^{s*}(I_t^a) \le \hat{x}_t^{s*}(I_t^a) = I_t^a$. The first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^{s*}(I_t^a) - d_t^{s*}(I_t) - \gamma(I_t^a)) = -(\beta^s + \theta) \le \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0)$. If $d_t^{s*}(I_t^a) < \hat{d}_t^{s*}(I_t^a)$, Lemma EC.1 implies that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \le \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a)$. Hence,

$$\begin{aligned}\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) =& \partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) + \partial_X H_t^s(x_t^{s*}(I_t^a) - d_t^{s*}(I_t) - \gamma(I_t^a)) \\ \le& \partial_{d_t} \hat{J}_t^s(I_t^a, \hat{d}_t^{s*}(I_t^a), I_t^a) + \partial_X \hat{H}_t^s(I_t^a - \hat{d}_t^{s*}(I_t^a) - \gamma_0) \\ =& \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a).\end{aligned} \tag{EC.8}$$

The strict concavity of $R^s(\cdot, I_t^a)$ and the supermodularity of $R^s(\cdot,\cdot)$ imply that

$$\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) > \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) \ge \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), -\infty) = \partial_{d_t} \hat{R}^s(\hat{d}_t^{s*}(I_t^a), I_t^a),$$

which contradicts inequality (EC.8). Hence, $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$. Finally, if $I_t^a \ge \hat{I}_t^H$, $d_t^{s*}(I_t^a) = d_t^H(I_t^a) \ge \hat{d}_t^H(I_t^a) = \hat{d}_t^{s*}(I_t^a)$. We have completed the proof of $d_t^{s*}(I_t^a) \ge \hat{d}_t^{s*}(I_t^a)$ for all $I_t^a \le K_a$.

To complete the induction, it suffices to show that if $\partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a) \le \partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a)$, for all $I_t^a \le K_a$. Note that $\hat{x}_t^{d*}(I_t)$ and $\hat{d}_t^{s*}(I_t^a)$ are constant if $I_t^a \le \hat{I}_t^L$ and $I_t \ge \hat{I}_t^H$, by Theorem 1 in Federgruen and Heching (1999). Hence, $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \le \hat{I}_t^L$ and $I_t^a \ge \hat{I}_t^H$, since $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a) = c$, if $I_t^a \le \hat{I}_t^L$, and $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a) = c - \theta = s$, if $I_t^a \ge \hat{I}_t^H$. If $\hat{I}_t^L \le I_t \le \hat{I}_t^H$, there are two possible cases: $\hat{I}_t^L \le I_t^H \le \hat{I}_t^H$ and $I_t^H \le \hat{I}_t^L \le \hat{I}_t^H$.

If $I_t^H \le \hat{I}_t^L$, $\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a)$ for all $I_t^a \le K_a$ follows immediately. Now assume that $I_t^H \in [\hat{I}_t^L, \hat{I}_t^H]$. If $I_t \in [\hat{I}_t^L, I_t^H]$, $x_t^{s*}(I_t^a) = \hat{x}_t^{s*}(I_t^a) = I_t^a$. Hence,

$$\begin{cases}\partial_{I_t^a} V_t^s(I_t^a) = & c + \beta^s + \partial_{I_t^a} R^s(d_t^{s*}(I_t^a), I_t^a) + \partial_X H_t^s(I_t^a - y_t^s(I_t^a)) - \gamma'(I_t^a)\partial_X H_t^s(I_t^a - y_t^s(I_t^a)), \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + \beta^s + \partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)),\end{cases}$$

where $y_t^s(I_t^a) = d_t^{s*}(I_t^a) + \gamma(I_t^a)$ and $\hat{y}_t^s(I_t^a) = \hat{d}_t^{s*}(I_t^a) + \gamma_0$. It suffices to show that $\partial_X H_t^s(I_t^a - y_t^s(I_t^a)) \le \partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a))$. We use the following lemma to prove this inequality:

LEMMA EC.5. *Let* $y_1 = \arg\max_{y_t}\{R^{s*}(y_t, \gamma_0) + \hat{H}_t^s(I_t^a - y_t)\}$, $y_2 = \arg\max_{y_t}\{R^{s*}(y_t, \gamma_0) + H_t^s(I_t^a - y_t)\}$ *and* $y_3 = \arg\max_{y_t}\{R^{s*}(y_t, \gamma(I_t^a)) + H_t^s(I_t^a - y_t)\}$, *for* $I_t^a \in [\hat{I}_t^L, I_t^H]$. *We have* $\partial_X \hat{H}_t^s(I_t^a - y_1) \ge \partial_X H_t^s(I_t^a - y_2) \ge \partial_X H_t^s(I_t^a - y_3)$.

Proof of Lemma EC.5: Since $\partial_X \hat{H}^s_t(X) \geq \partial_X H^s_t(X)$, $\partial_{y_t} R^{s*}(y_1,\gamma_0) - \partial_X H^s_t(I^a_t - y_1) \geq \partial_{y_t} R^{s*}(y_1,\gamma_0) - \partial_X \hat{H}^s_t(I^a_t - y_1)$, i.e., $y_1 \leq y_2$. If $y_1 = y_2$, $\partial_X \hat{H}^s_t(I^a_t - y_1) \geq \partial_X H^s_t(I^a_t - y_2)$ follows from $\partial_X \hat{H}^s_t(X) \geq \partial_X H^s_t(X)$ for any X. If $y_1 < y_2$, $\partial_{y_t} R^{s*}(y_1,\gamma_0) > \partial_{y_t} R^{s*}(y_2,\gamma_0)$ by the strict concavity of $R^{s*}(\cdot,\cdot)$, and $\partial_{y_t} R^{s*}(y_1,\gamma_0) - \partial_X \hat{H}^s_t(I^a_t - y_1) \leq \partial_{y_t} R^{s*}(y_2,\gamma_0) - \partial_X H^s_t(I^a_t - y_2)$ by Lemma EC.1. Hence, $\partial_X \hat{H}^s_t(I^a_t - y_1) > \partial_X H^s_t(I^a_t - y_2)$. For the second inequality, the supermodularity of $R^{s*}(\cdot,\cdot)$ yields that $y_2 \geq y_3$ and, thus, $\partial_X H^s_t(I^a_t - y_2) \geq \partial_X H^s_t(I^a_t - y_3)$. *Q.E.D.*

Invoking Lemma EC.5,

$$\partial_X H^s_t(I^a_t - y^s_t(I^a_t)) = \partial_X H^s_t(I^a_t - y_3) \leq \partial_X \hat{H}^s_t(I^a_t - y_1) = \partial_X \hat{H}^s_t(I^a_t - \hat{y}^s_t(I^a_t)).$$

Hence, $\partial_{I^a_t} V^s_t(I^a_t) \leq \partial_{I^a_t} \hat{V}^s_t(I^a_t)$ for all $I^a_t \in [\hat{I}^L_t, I^H_t]$. If $I^a_t \in [I^H_t, \hat{I}^H_t]$,

$$\partial_{I^a_t} V^s_t(I^a_t) \leq c - \theta = \partial_{I^a_t} \hat{V}^s_t(\hat{I}^H_t) \leq \partial_{I^a_t} \hat{V}^s_t(I^a_t),$$

where the first inequality follows from the first order condition with respect to x^a_t. This completes the induction and the proof of **part (a)**.

To prove **part (b)**, it suffices to show that if $\partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1}) \leq \partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1})$ for all $I^a_{t-1} \leq K_a$, (a) $x^L_t(I^a_t) \leq \hat{x}^L_t(I^a_t)$, (b) $x^H_t(I^a_t) \leq \hat{x}^H_t(I^a_t)$, and (c) $\partial_{I^a_t} V^s_t(I^a_t) \leq \partial_{I^a_t} \hat{V}^s_t(I^a_t)$, for all $I^a_t \leq K_a$. For $t = 0$, $\partial_{I^a_0} V^s_0(I^a_0) = \partial_{I^a_0} \hat{V}^s_0(I^a_0) = 0$ for $I^a_0 \leq K_a$.

First, we show that $x^L_t(I^a_t) \leq \hat{x}^L_t(I^a_t)$, and the proof of $x^H_t(I^a_t) \leq \hat{x}^H_t(I^a_t)$ follows from the same argument. If $x^L_t(I^a_t) > \hat{x}^L_t(I^a_t)$, Lemma EC.1 yields that $\partial_{x^a_t} J^L_t(x^L_t(I^a_t), d_0, I^a_t) \geq \partial_{x^a_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), d_0, I^a_t)$. Hence,

$$\partial_X H^s_t(x^L_t(I^a_t) - y^L_t(I^a_t)) = \partial_{x^a_t} J^L_t(x^L_t(I^a_t), d_0, I^a_t) - \beta^s \geq \partial_{x^a_t} \hat{J}^L_t(\hat{x}^L_t(I^a_t), d_0, I^a_t) - \beta^s = \partial_X \hat{H}^s_t(\hat{x}^L_t(I^a_t) - \hat{y}^L_t(I^a_t)).$$

Since $\partial_X H^s_t(X) \leq \partial_X \hat{H}^s_t(X)$ for any X and both of them are strictly decreasing, $y^L_t(I^a_t) > \hat{y}^L_t(I^a_t)$. However, $y^L_t(I^a_t) = d_0 + \gamma(I^a_t) \leq d_0 + \hat{\gamma}(I^a_t) = \hat{y}^L_t(I^a_t)$. This contradiction shows that $x^L_t(I^a_t) \leq \hat{x}^L_t(I^a_t)$. $x^H_t(I^a_t) \leq \hat{x}^H_t(I^a_t)$ follows analogously.

To complete the proof, we need to show $\partial_{I^a_t} V^s_t(I^a_t) \leq \partial_{I^a_t} \hat{V}^s_t(I^a_t)$ for all $I^a_t \leq K_a$. For the case $I^a_t \in [\hat{I}^L_t, \hat{I}^H_t]$, the proof is identical to that of part (a), and, hence, omitted. If $I^a_t \leq I^L_t$,

$$\begin{cases} \partial_{I^a_t} V^s_t(I^a_t) = & c + (p_0 - b - \alpha c)\gamma'(I^a_t) - \gamma'(I^a_t)\partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)), \\ \partial_{I^a_t} \hat{V}^s_t(I^a_t) = & c + (p_0 - b - \alpha c)\hat{\gamma}'(I^a_t) - \hat{\gamma}'(I^a_t)\partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t)). \end{cases}$$

Since $x^{s*}_t(I^a_t) \leq \hat{x}^{s*}_t(I^a_t)$, there are two cases: (a) $x^{s*}_t(I^a_t) = \hat{x}^{s*}_t(I^a_t)$ and (b) $x^{s*}_t(I^a_t) < \hat{x}^{s*}_t(I^a_t)$. If $x^{s*}_t(I^a_t) = \hat{x}^{s*}_t(I^a_t)$, $x^{s*}_t(I^a_t) - y^s_t(I^a_t) \geq \hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t)$ and, hence, $\partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)) \leq \partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t))$, since $\partial_X H^s_t(X) \leq \partial_X \hat{H}^s_t(X)$ for any X. If $x^{s*}_t(I^a_t) < \hat{x}^{s*}_t(I^a_t)$, Lemma EC.1 yields that $\partial_{x^a_t} J^s_t(x^{s*}_t(I^a_t), d_0, I^a_t) \leq \partial_{x^a_t} \hat{J}^s_t(\hat{x}^{s*}_t(I^a_t), d_0, I^a_t)$. Hence,

$$\partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)) = \partial_{x^a_t} J^s_t(x^{s*}_t(I^a_t), d_0, I^a_t) - \beta^s \leq \partial_{x^a_t} \hat{J}^s_t(\hat{x}^{s*}_t(I^a_t), d_0, I^a_t) - \beta^s = \partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t)).$$

We have thus showed that $\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a)) \leq \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))$ in both cases. Therefore,

$$\begin{aligned}\partial_{I_t^a} V_t^s(I_t^a) =& c+\gamma'(I_t^a)(p_0-b-\alpha c-\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a)))\\ \leq& c+\hat{\gamma}'(I_t^a)(p_0-b-\alpha c-\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))\\ =&\partial_{I_t^a}\hat{V}_t^s(I_t^a),\end{aligned}$$

where the inequality follows from $\gamma'(I_t^a)\leq\hat{\gamma}'(I_t^a)\leq 0$ and

$$p_0-b-\alpha c-\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))\geq p_0-b-\alpha c-\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))>0.$$

The proof of the case $I_t^a\geq \hat{I}_t^H$ follows from the identical argument of the case $I_t^a\leq I_t^L$, and is, hence, omitted.

If $I_t^a\in[I_t^L,\hat{I}_t^L]$,

$$\begin{cases}\partial_{I_t^a} V_t^s(I_t^a)= & c+\beta^s+(p_0-b-\alpha c)\gamma'(I_t^a)+\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))-\gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a)),\\ \partial_{I_t^a} \hat{V}_t^s(I_t^a)= & c+(p_0-b-\alpha c)\hat{\gamma}'(I_t^a)-\hat{\gamma}'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a)).\end{cases}$$

Note that $\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))\leq-\beta^s=\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))$. Therefore,

$$\begin{aligned}\partial_{I_t^a} V_t^s(I_t^a) =& c+\beta^s+\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))+\gamma'(I_t^a)(p_0-b-\alpha c-\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a)))\\ \leq& c+\hat{\gamma}'(I_t^a)(p_0-b-\alpha c-\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a)))\\ =&\partial_{I_t^a}\hat{V}_t^s(I_t^a),\end{aligned} \tag{EC.9}$$

where the inequality follows from $\gamma'(I_t^a)\leq\hat{\gamma}'(I_t^a)\leq 0$, $\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))\leq\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))$, and

$$p_0-b-\alpha c-\partial_X H_t^s(x_t^{s*}(I_t^a)-y_t^s(I_t^a))\geq p_0-b-\alpha c-\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a)-\hat{y}_t^s(I_t^a))>0.$$

We have thus showed $\partial_{I_t^a} V_t^s(I_t^a)\leq\partial_{I_t^a}\hat{V}_t^s(I_t^a)$ for all $I_t^a\leq K_a$, which completes the proof of **part (b)**. *Q.E.D.*

Proof of Theorem 5: We employ backward induction to prove **parts (a) - (d)** together. We define $H_t^s(X):=\mathbb{E}_{\epsilon_t^a}\{-(b+h_a)(X-\epsilon_t^a)^+ +\alpha(V_{t-1}^s(X)-cX)\}$ and $\hat{H}_t^s(X):=\mathbb{E}_{\epsilon_t^a}\{-(b+h_a)(X-\epsilon_t^a)^+ +\alpha(\hat{V}_{t-1}^s(X)-cX)\}$, so that $g_t^s(x_t^a,d_t,I_t^a):=H_t^s(x_t^a-d_t-\gamma(I_t^a))$ and $\hat{g}_t^s(x_t^a,d_t,I_t^a):=\hat{H}_t^s(x_t^a-d_t-\gamma(I_t^a))$. We define the objective functions $J_t^L(x_t^a,d_t,I_t^a):=R^s(d_t,I_t^a)+\beta^s x_t^a+g_t^s(x_t^a,d_t,I_t^a)$, $\hat{J}_t^L(x_t^a,d_t,I_t^a):=\hat{R}^s(d_t,I_t^a)+\beta^s x_t^a+\hat{g}_t^s(x_t^a,d_t,I_t^a)$, $J_t^H(x_t^a,d_t,I_t^a):=R^s(d_t,I_t^a)+(\beta^s+\theta)x_t^a+g_t^s(x_t^a,d_t,I_t^a)$, and $\hat{J}_t^H(x_t^a,d_t,I_t^a):=\hat{R}^s(d_t,I_t^a)+(\beta^s+\hat{\theta})x_t^a+\hat{g}_t^s(x_t^a,d_t,I_t^a)$, where $\hat{\theta}=c-\hat{s}\leq c-s=\theta$. Let $\gamma_t:=\gamma(I_t^a)$, $y_t^s(I_t^a):=d_t^{s*}(I_t^a)+\gamma_t$ and $\hat{y}_t^s(I_t^a):=\hat{d}_t^{s*}(I_t^a)+\gamma_t$.

It suffices to show that if $\partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$ for all $I^a_{t-1} \leq K_a$, (1) $\hat{I}^L_t \geq I^L_t$, (2) $\hat{x}^{s*}_t(I^a_t) \geq x^{s*}_t(I^a_t)$ for all $I^a_t \leq \hat{I}^H_t$, (3) $\hat{d}^{s*}_t(I^a_t) \leq d^{s*}_t(I^a_t)$, and (4) $\partial_{I^a_t} \hat{V}^s_t(I^a_t) \geq \partial_{I^a_t} V^s_t(I^a_t)$. Since $\partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, $\partial_X \hat{H}^s_t(X) \geq \partial_X H^s_t(X)$. For $t=0$, $\partial_{I^a_0} \hat{V}^s_0(I^a_0) = \partial_{I^a_0} V^s_0(I^a_0) = 0$, so the initial condition is satisfied.

We first show that if $\partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, $\hat{x}^L_t(I^a_t) \geq x^L_t(I^a_t)$, $\hat{d}^L_t(I^a_t) \leq d^L_t(I^a_t)$, and $\hat{d}^H_t(I^a_t) \leq d^H_t(I^a_t)$. $\hat{x}^L_t(I^a_t) \geq x^L_t(I^a_t)$ and $\hat{d}^L_t(I^a_t) \leq d^L_t(I^a_t)$ follows from the same argument as the proof of Theorem 4. We show by contradiction that $\hat{d}^H_t(I^a_t) \leq d^H_t(I^a_t)$.

Assume that $d^H_t(I^a_t) < \hat{d}^H_t(I^a_t)$, so $y^H_t(I^a_t) = d^H_t(I^a_t) + \gamma_t < \hat{d}^H_t(I^a_t) + \gamma_t = \hat{y}^H_t(I^a_t)$. Lemma EC.1 yields that $\partial_{d_t} J^H_t(x^H_t(I^a_t), d^H_t(I^a_t), I^a_t) \leq \partial_{d_t} \hat{J}^H_t(\hat{x}^H_t(I^a_t), \hat{d}^H_t(I^a_t), I^a_t)$. The strict concavity of $R^s(\cdot, I^a_t)$ imply that $\partial_{d_t} R^s(d^H_t(I^a_t), I^a_t) > \partial_{d_t} R^s(\hat{d}^H_t(I^a_t), I^a_t)$. Hence, we have:

$$
\begin{aligned}
\partial_X H^s_t(x^H_t(I^a_t) - y^H_t(I^a_t)) =& \partial_{d_t} R^s(d^H_t(I^a_t), I^a_t) - \partial_{d_t} J^H_t(x^H_t(I^a_t), d^H_t(I^a_t), I^a_t) \\
>& \partial_{d_t} \hat{R}^s(\hat{d}^H_t(I^a_t), I^a_t) - \partial_{d_t} \hat{J}^H_t(\hat{x}^H_t(I^a_t), \hat{d}^H_t(I^a_t), I^a_t) \\
=& \partial_X \hat{H}^s_t(\hat{x}^H_t(I^a_t) - \hat{y}^H_t(I^a_t)).
\end{aligned}
$$

The first order condition with respect to x^a_t implies that

$$
\partial_X H^s_t(x^H_t(I^a_t) - y^H_t(I^a_t)) = -(\beta^s + \theta) < -(\beta^s + \hat{\theta}) = \partial_X \hat{H}^s_t(\hat{x}^H_t(I^a_t) - \hat{y}^H_t(I^a_t)),
$$

which leads to a contradiction. Hence, $d^H_t(I^a_t) \geq \hat{d}^H_t(I^a_t)$. We have thus proved that, if $\partial_{I^a_{t-1}} \hat{V}^s_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, $\hat{x}^L_t(I^a_t) \geq x^L_t(I^a_t)$, $\hat{d}^L_t(I^a_t) \leq d^L_t(I^a_t)$, and $\hat{d}^H_t(I^a_t) \leq d^H_t(I^a_t)$.

Next, we show that $\hat{d}^{s*}_t(I^a_t) \leq d^{s*}_t(I^a_t)$ for all $I^a_t \leq K_a$. If $I^a_t \leq I^L_t$ or $I^a_t \geq \max\{I^H_t, \hat{I}^H_t\}$, $\hat{d}^{s*}_t(I^a_t) \leq d^{s*}_t(I^a_t)$ follows from $\hat{d}^L_t(I^a_t) \leq d^L_t(I^a_t)$ and $\hat{d}^H_t(I^a_t) \leq d^H_t(I^a_t)$. Now we assume that $I^a_t \in [I^L_t, \max\{I^H_t, \hat{I}^H_t\}]$. If $I^a_t \in [I^L_t, \hat{I}^L_t]$, $x^{s*}_t(I^a_t) = I^a_t \leq \hat{x}^{s*}_t(I^a_t)$. If $\hat{d}^{s*}_t(I^a_t) > d^{s*}_t(I^a_t)$, by Lemma EC.1, $\partial_{d_t} J^s_t(x^{s*}_t(I^a_t), d^{s*}_t(I^a_t), I^a_t) \leq \partial_{d_t} \hat{J}^s_t(\hat{x}^{s*}_t(I^a_t), \hat{d}^{s*}_t(I^a_t), I^a_t)$. The first order condition with respect to x^a_t implies that $\partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)) \leq -\beta^s = \partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t))$. Therefore,

$$
\begin{aligned}
\partial_{d_t} R^s(d^{s*}_t(I^a_t), I^a_t) =& \partial_{d_t} J^s_t(x^{s*}_t(I^a_t), d^{s*}_t(I^a_t), I^a_t) + \partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)) \\
\leq& \partial_{d_t} \hat{J}^s_t(\hat{x}^{s*}_t(I^a_t), \hat{d}^{s*}_t(I^a_t), I^a_t) + \partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t)) \\
=& \partial_{d_t} R^s(\hat{d}^{s*}_t(I^a_t), I^a_t).
\end{aligned}
$$

However, $\hat{d}^{s*}_t(I^a_t) > d^{s*}_t(I^a_t)$ implies that $\partial_{d_t} R^s(d^{s*}_t(I^a_t), I^a_t) > \partial_{d_t} R^s(\hat{d}^{s*}_t(I^a_t), I^a_t)$. The contradiction shows that if $I^a_t \in [I^L_t, \hat{I}^L_t]$, $\hat{d}^{s*}_t(I^a_t) \leq d^{s*}_t(I^a_t)$.

If $I^a_t \in [\hat{I}^L_t, I^H_t]$, $x^{s*}_t(I^a_t) = I^a_t \geq \hat{x}^{s*}_t(I^a_t)$. If $\hat{d}^{s*}_t(I^a_t) > d^{s*}_t(I^a_t)$, Lemma EC.1 implies that $\partial_{d_t} J^s_t(x^{s*}_t(I^a_t), d^{s*}_t(I^a_t), I^a_t) \leq \partial_{d_t} \hat{J}^s_t(\hat{x}^{s*}_t(I^a_t), \hat{d}^{s*}_t(I^a_t), I^a_t)$. Since $\partial_X H^s_t(X) \leq \partial_X \hat{H}^s_t(X)$ for any X and $\hat{d}^{s*}_t(I^a_t) > d^{s*}_t(I^a_t)$, $\partial_X H^s_t(x^{s*}_t(I^a_t) - y^s_t(I^a_t)) \leq \partial_X \hat{H}^s_t(\hat{x}^{s*}_t(I^a_t) - \hat{y}^s_t(I^a_t))$. We apply the same argument as in the case $I^a_t \in [I^L_t, \hat{I}^L_t]$ and the contradiction shows that $\hat{d}^{s*}_t(I^a_t) \leq d^{s*}_t(I^a_t)$ for all $I^a_t \in [\hat{I}^L_t, I^H_t]$.

If $I_t^a \in [I_t^H, \hat{I}_t^H]$ (which might be an empty set), the first order condition with respect to x_t^a implies that

$$\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^{s*}(I_t^a)) = -(\beta^s + \theta) < -(\beta^s + \hat{\theta}) \le \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^{s*}(I_t^a)). \tag{EC.10}$$

If $\hat{d}_t^{s*}(I_t^a) > d_t^{s*}(I_t^a)$, Lemma EC.1 implies that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \ge \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$. The same argument as in the case $I_t^a \in [\hat{I}_t^L, I_t^H]$ proves that $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$ for all $I_t^a \in [I_t^H, \hat{I}_t^H]$. Hence, $\hat{d}_t^{s*}(I_t^a) \le d_t^{s*}(I_t^a)$ for all $I_t^a \le K_a$.

To complete the induction, we next show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \le K_a$.

If $I_t^a \le I_t^L$, note that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) = \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$ and $\partial_{d_t} \hat{J}_t^d(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a) = \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. By the first order condition with respect to x_t^a, $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) = -\beta^s$. A simple contradiction argument leads to that $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$, for $I_t^a \le I_t^L$. Therefore:

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Hence, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) = \partial_{I_t^a} V_t^s(I_t^a)$, for $I_t^a \le I_t^L$.

If $I_t^a \in [I_t^L, \hat{I}_t^L]$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1 - \gamma'(I_t^a))\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

Note that the first order condition with respect to x_t^a implies that $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \le -\beta^s = \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a))$. If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$,

$$\begin{aligned} &\partial_{I_t^a} \hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \\ =& -\gamma'(I_t^a)(\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) + \beta^s + \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \ge 0. \end{aligned}$$

If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, Lemma EC.1 yields that $\partial_{d_t} J_t^s(x_t^{s*}(I_t^a), d_t^{s*}(I_t^a), I_t^a) \ge \partial_{d_t} \hat{J}_t^s(\hat{x}_t^{s*}(I_t^a), \hat{d}_t^{s*}(I_t^a), I_t^a)$, i.e.,

$$\partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \ge \partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \tag{EC.11}$$

We have:

$$\begin{aligned} \partial_{I_t^a} \hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) =& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)))]\gamma'(I_t^a) \\ & - (\beta^s + \partial_X H_t^s(x_t^{s*}(I^a) - y_t^s(I_t^a))) \\ \ge& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)))]\gamma'(I_t^a) \\ \ge& [(p(\hat{d}_t^{s*}(I_t^a)) - p(d_t^{s*}(I_t^a))) - (\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a))]\gamma'(I_t^a) \\ =& [p'(d_t^{s*}(I_t^a)) y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a)) \hat{y}_t^s(I_t^a)]\gamma'(I_t^a) \\ \ge& 0, \end{aligned} \tag{EC.12}$$

where the first inequality follows from $\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) + \beta^s \le 0$, the second inequality from (EC.11), and the last from the concavity of $p(\cdot)$ and $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$.

If $I_t^a \in [\hat{I}_t^L, I_t^H]$, $x_t^{s*}(I_t^a) = I_t^a \ge \hat{x}_t^{s*}(I_t^a)$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) & = c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) & = c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$, $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) \ge \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$, and $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$. If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, as in (EC.12),

$$\begin{aligned} \partial_{I_t^a} \hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \ge & [p'(d_t^{s*}(I_t^a)) y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a)) \hat{y}_t^s(I_t^a)]\gamma'(I_t^a) \\ & + (\partial_{d_t} R^s(\hat{d}_t^{s*}(I_t^a), I_t^a) - \partial_{d_t} R^s(d_t^{s*}(I_t^a), I_t^a)) \\ > & 0, \end{aligned} \tag{EC.13}$$

where the second inequality follows from $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$.

If $I_t^a \in [I_t^H, \hat{I}_t^H]$ (which might be an empty set),

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) - \theta \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) + \beta^s + (1-\gamma'(I_t^a))\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)). \end{cases}$$

(EC.10) implies that $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) > \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))$ and $\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) + \beta^s + \theta \ge 0$. The same argument as in the case $I_t^a \in [I_t^L, \hat{I}_t^L]$ implies that $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$ for $I_t^a \in [I_t^H, \hat{I}_t^H]$.

If $I_t^a \ge \max\{I_t^H, \hat{I}_t^H\}$,

$$\begin{cases} \partial_{I_t^a} V_t^s(I_t^a) = & c + (p(d_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)) - \theta \\ \partial_{I_t^a} \hat{V}_t^s(I_t^a) = & c + (p(\hat{d}_t^{s*}(I_t^a)) - b - \alpha c)\gamma'(I_t^a) - \gamma'(I_t^a)\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \hat{\theta}. \end{cases}$$

Note that

$$\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) = -(\beta^s + \hat{\theta}) \ge -(\beta^s + \theta) = \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a)).$$

If $d_t^{s*}(I_t^a) = \hat{d}_t^{s*}(I_t^a)$,

$$\hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) = -\gamma'(I_t^a)(\partial_X \hat{H}_t^s(\hat{x}_t^{s*}(I_t^a) - \hat{y}_t^s(I_t^a)) - \partial_X H_t^s(x_t^{s*}(I_t^a) - y_t^s(I_t^a))) + \theta - \hat{\theta} > 0.$$

If $d_t^{s*}(I_t^a) > \hat{d}_t^{s*}(I_t^a)$, the same argument as (EC.12) implies that

$$\hat{V}_t^s(I_t^a) - \partial_{I_t^a} V_t^s(I_t^a) \ge [p'(d_t^{s*}(I_t^a)) y_t^s(I_t^a) - p'(\hat{d}_t^{s*}(I_t^a)) \hat{y}_t^s(I_t^a)]\gamma'(I_t^a) + \theta - \hat{\theta} > 0.$$

We have thus showed that, if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \ge \partial_{I_{t-1}^a} V_{t-1}^s(I_{t-1}^a)$ for all $I_{t-1}^a \le K_a$, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \le K_a$, which completes the proof of **Theorem 5**. *Q.E.D.*

Proof of Theorem 6: We first show **part (a)**. Observe that if $h_w \geq h_a$ and $\hat{\gamma}(I_t^a) \equiv \gamma_0$ for all $I_t^a \leq K_a$, withholding positive inventory is dominated by displaying this part of inventory to customers, because the holding cost at the customer-accessible storage is smaller than that at the warehouse, and there is no demand-suppressing effect of customer-accessible inventory. Therefore, the firm should not withhold any inventory if $h_w \geq h_a$ and $\hat{\gamma}(I_t^a) \equiv \gamma_0$ for all $I_t^a \leq K_a$.

Next, we show **part (b)** by backward induction. Since it is optimal for the firm not to withhold any inventory in the model with demand $\hat{D}_t$, this model is reduced to the one discussed in Section 5.1, i.e., the model without inventory withholding. Let $K_t(I_t^a) := V_t(I_t^a, I_t^a)$. It suffices to show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a)$, for all $I_{t-1}^a \leq K_a$, (a) $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$, (b) $d_t(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$, and (c) $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} K_t(I_t^a)$, for all $I_t^a \leq K_a$. For $t = 0$, $\hat{V}_0^s(I_0^a) = K_0(I_0^a) = 0$, so the initial condition is satisfied.

If $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_t(I_{t-1}^a)$ for $I_{t-1}^a \leq K_a$,

$$\partial_X \hat{H}_t^s(X) \geq \partial_X L_t(X, Y) + \partial_Y L_t(X, Y) \text{ for } X = Y,$$

where $\hat{H}_t^s(X)$ is defined in the proof of Theorem 4, and

$$L_t(X, Y) := \mathbb{E}_{\epsilon_t^a}\{-(h_a + b)(X - \epsilon_t^a)^+ + \alpha[V_{t-1}(X - \epsilon_t^a, Y - \epsilon_t^a) - cY]\}.$$

Therefore, the same argument as in the proof of Theorem 4(a) shows that $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$ and $d_t(I_t^a) \geq \hat{d}_t^{s*}(I_t^a)$.

To complete the induction, we show that if $\partial_{I_{t-1}^a} \hat{V}_{t-1}^s(I_{t-1}^a) \geq \partial_{I_{t-1}^a} K_{t-1}(I_{t-1}^a)$ for all $I_{t-1}^a \leq K_a$, $\partial_{I_t^a} \hat{V}_t^s(I_t^a) \geq \partial_{I_t^a} K_t(I_t^a)$, for $I_t^a \leq K_a$. Since $x_t^a(I_t^a) \leq \hat{x}_t^{s*}(I_t^a)$, $x_t^a(I_t^L) \leq x_t^{s*}(I_t^L) = I_t^L$. If $I_t^a \leq I_t^L$,

$$\partial_{I_t^a} K_t(I_t^a) \leq c + (\underline{p} - b - \alpha c)\gamma'(I_t^a) \leq c = \partial_{I_t^a} \hat{V}_t^s(I_t^a).$$

For the case $I_t^a \geq I_t^L$, the argument is very similar to that in the proof of Theorem 4, so we only sketch it. The key step is to show that

$$\partial_X \hat{H}_t^s(I_t^a - \hat{y}_t^s(I_t^a)) \geq \partial_X L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)) + \partial_Y L_t(x_t^{a*}(I_t^a, I_t^a) - y_t(I_t^a), I_t^a - y_t(I_t^a)),$$

where $\hat{y}_t^s(I_t^a)$ is defined in the proof of Theorem 4 and $y_t(I_t^a) := d_t^{a*}(I_t^a, I_t^a) + \gamma(I_t^a)$. To show the above inequality, let $y_t^*(I_t^a)$ be the optimal expected demand in the system with demand $\hat{D}_t$ such that the firm is forced to display $x_t^{a*}(I_t^a, I_t^a)$ to customers and withhold $I_t^a - x_t^{a*}(I_t^a, I_t^a) > 0$ in the warehouse, when the current customer-accessible inventory level is $I_t^a > I_t^L$. Let

$$\hat{L}_t^s(X, Y) = \mathbb{E}_{\epsilon_t^a}\{-(h_a + b)(X - \epsilon_t^a)^+ + \alpha[\hat{V}_{t-1}^s(Y - \epsilon_t^a) - cY]\},$$

Following the same argument as the proof of Lemma EC.5, we have:

$$\begin{aligned}\partial_X \hat{H}^s_t(I^a_t - \hat{y}^s_t(I^a_t)) \geq& \partial_X \hat{L}^s_t(x^{a*}_t(I^a_t, I^a_t) - y^*_t(I^a_t), I^a_t - y^*_t(I^a_t)) + \partial_Y \hat{L}_t(x^{a*}_t(I^a_t, I^a_t) - y^*_t(I^a_t), I^a_t - y^*_t(I^a_t)) \\ \geq& \partial_X L_t(x^{a*}_t(I^a_t, I^a_t) - y_t(I^a_t), I^a_t - y_t(I^a_t)) + \partial_Y L_t(x^{a*}_t(I^a_t, I^a_t) - y_t(I^a_t), I^a_t - y_t(I^a_t)).\end{aligned} \tag{EC.14}$$

Based on (EC.14), the same argument as the proof of Theorem 4(a) yields that $\partial_{I^a_t} \hat{V}^s_t(I^a_t) \geq \partial_{I^a_t} K_t(I^a_t)$, for all $I^a_t \leq K_a$. This completes the induction and the proof of **Theorem 6(b)**. *Q.E.D.*

Proof of Theorem 7: We prove Theorem 7 by backward induction. Let $L_t(X,Y) := \mathbb{E}_{\epsilon^a_t}\{G_t(X - \epsilon^a_t, Y - \epsilon^a_t)\}$ and $H_t(X) := L_t(X,X)$, then $g^a_t(x^a_t, d_t, I^a_t) = H_t(x^a_t - d_t - \gamma(I^a_t))$. Let $K_t(I^a_t) = V_t(I^a_t, I^a_t)$.

It suffices to show that if $\partial_{I^a_{t-1}} K_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, for any $I^a_{t-1} \leq K_a$, (a) $x^a_t(I^a_t) \geq x^{s*}_t(I^a_t)$, (b) $d^*_t(I^a_t, I_t) \leq d^{s*}_t(I^a_t)$, and (c) $\partial_{I^a_t} K_t(I^a_t) \geq \partial_{I^a_t} V^s_t(I^a_t)$, for any $I^a_t \leq K_a$. For $t = 0$, $V^s_0(I^a_0) = K_0(I^a_0) = 0$, so the initial condition is satisfied. Because $\partial_{I^a_{t-1}} K_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, for any $I^a_{t-1} \leq K_a$, $\partial_X H_t(X) \geq \partial_X H^s_t(X)$ for any X.

Following the same argument as the proof of Theorem 5, we have that if $\partial_X H_t(X) \geq \partial_X H^s_t(X)$ for any X, $x^a_t(I^a_t) \geq x^L_t(I^a_t)$ and $d_t(I^a_t) \leq d^L_t(I^a_t)$. Hence, $I^L_t \leq I^*_t := \sup\{I^a_t : x^a_t(I^a_t) > I^a_t\}$. Therefore, we have that

$$d^*_t(I^a_t, I_t) = d_t(I^a_t) \leq d^L_t(I^a_t) \leq d^{s*}_t(I^a_t), \text{ if } I^a_t \leq I^*_t,$$

where the last inequality follows from the supermodularity of $J^s_t(x^a_t, d_t, I^a_t)$ in (x^a_t, d_t) for any fixed I^a_t.

If $I_t = I^a_t > I^*_t$, $x^{a*}_t(I^a_t, I_t) < x^*_t(I^a_t, I_t) = x^{s*}_t(I^a_t) = I^a_t = I_t$. Therefore,

$$d^{s*}_t(I^a_t) = \arg\max_{d_t \in [\underline{d}, \bar{d}]}\{R(d_t, I^a_t) + H^s_t(I^a_t - d_t - \gamma(I^a_t))\} \geq \hat{d}_t(I^a_t) := \arg\max_{d_t \in [\underline{d}, \bar{d}]}\{R(d_t, I^a_t) + H_t(I^a_t - d_t - \gamma(I^a_t))\},$$

since

$$\partial_{d_t} R(\hat{d}_t(I^a_t), I^a_t) - \partial_X H^s_t(I^a_t - \hat{d}_t(I^a_t) - \gamma(I^a_t)) \geq \partial_{d_t} R(\hat{d}_t(I^a_t), I^a_t) - \partial_X H_t(I^a_t - \hat{d}_t(I^a_t) - \gamma(I^a_t)),$$

where the inequality follows from $\partial_X H_t(X) \geq \partial_X H^s_t(X)$ for all X. Similar argument yields that:

$$\begin{aligned} d^*_t(I^a_t, I_t) =& \arg\max_{d_t \in [\underline{d}, \bar{d}]}\{R(d_t, I^a_t) + L_t(x^{a*}_t(I^a_t, I_t) - d_t - \gamma(I^a_t), I^a_t - d_t - \gamma(I^a_t))\} \\ \leq& \hat{d}_t(I^a_t) = \arg\max_{d_t \in [\underline{d}, \bar{d}]}\{R(d_t, I^a_t) + L_t(I^a_t - d_t - \gamma(I^a_t), I^a_t - d_t - \gamma(I^a_t))\},\end{aligned}$$

because $L_t(\cdot, Y)$ is concave for any fixed Y. Hence, $d^*_t(I^a_t, I_t) \leq \hat{d}_t(I^a_t) \leq d^{s*}_t(I^a_t)$ for any $I_t = I^a_t \geq I^*_t$.

To complete the induction, we need to show that if $\partial_{I^a_{t-1}} K_{t-1}(I^a_{t-1}) \geq \partial_{I^a_{t-1}} V^s_{t-1}(I^a_{t-1})$, for any $I^a_{t-1} \leq K_a$, $\partial_{I^a_t} K_t(I^a_t) \geq \partial_{I^a_t} V^s_t(I^a_t)$, for any $I^a_t \leq K_a$. For $I^a_t \leq I^*_t$, $x^{a*}_t(I^a_t, I_t) = x^*_t(I^a_t, I_t)$. Same argument as in the proof of Theorem 5 implies that $\partial_{I^a_t} K_t(I^a_t) \geq \partial_{I^a_t} V^s_t(I^a_t)$, if $I^a_t \leq I^*_t$.

If $I^a_t > I^*_t$, the proof is based on the following lemma:

Lemma EC.6. *Assume that $I_t^a > I_t^*$. Let*

$$\hat{V}_t^s(I_t^a) = cI_t^a + \max_{d_t \in [\underline{d}, \bar{d}]} \{R(d_t, I_t^a) + \beta I_t^a + L_t(I_t^a - d_t - \gamma(I_t^a), I_t^a - d_t - \gamma(I_t^a))\}.$$

We have:

$$\partial_{I_t^a} V_t^s(I_t^a) \le \partial_{I_t^a} \hat{V}_t^s(I_t^a) \le \partial_{I_t^a} K_t(I_t^a). \tag{EC.15}$$

Proof of Lemma EC.6: The first inequality follows from the same argument as the proof of Theorem 5. For the second inequality, observe that

$$\begin{aligned}
\partial_{I_t^a} \hat{V}_t(I_t^a) =& c + \beta + (p(\hat{d}_t(I_t^a)) - b - \alpha c)\gamma'(I_t^a) \\
&+ (1-\gamma'(I_t^a))\partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\
&+ (1-\gamma'(I_t^a))\partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)),
\end{aligned}$$

$$\begin{aligned}
\text{and } \partial_{I_t^a} K_t(I_t^a) =& c + \beta + (p(d_t^*(I_t^a, I_t)) - b - \alpha c)\gamma'(I_t^a) \\
&+ (1-\gamma'(I_t^a))\partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\
&+ (1-\gamma'(I_t^a))\partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a))).
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_{I_t^a} K_t(I_t^a) - \partial_{I_t^a} \hat{V}_t(I_t^a) =& (p(d_t^*(I_t^a, I_t)) - p(\hat{d}_t(I_t^a)))\gamma'(I_t^a) \\
&- \gamma'(I_t^a)[\partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\
&- \partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\
&+ \partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\
&- \partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a))] \\
&+ \partial_X L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\
&- \partial_X L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)) \\
&+ \partial_Y L_t(x_t^{a*}(I_t^a, I_t) - d_t^*(I_t^a, I_t) - \gamma(I_t^a), I_t^a - d_t^*(I_t^a, I_t) - \gamma(I_t^a)) \\
&- \partial_Y L_t(I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a), I_t^a - \hat{d}_t(I_t^a) - \gamma(I_t^a)).
\end{aligned} \tag{EC.16}$$

Based on the first order condition with respect to d_t and Lemma EC.1, the same argument as inequality (EC.12) yields that $\partial_{I_t^a} K_t(I_t^a) - \partial_{I_t^a} \hat{V}_t(I_t^a) \ge 0$, and hence (EC.15) holds. *Q.E.D.*

By Lemma EC.6, $\partial_{I_t^a} K_t(I_t^a) \ge \partial_{I_t^a} \hat{V}_t^s(I_t^a) \ge \partial_{I_t^a} V_t^s(I_t^a)$ for all $I_t^a \le K_a$. This completes the induction in the proof of **Theorem 7**. *Q.E.D.*

Proof of Theorem 8: The proof, based on backward induction, is very similar to that of Lemma 4 and Theorem 1, so we only sketch it. In particular, the continuous differentiability of $V_t^r(I_t^a, I_t)$

follows from the same argument as in the proof of Lemma 4 and is, hence, omitted. Note that $V_0^r(I_0^r, I_0) - cI_0 - r_d I_0^a = -cI_0 - r_d I_0^a$ is jointly concave, continuously differentiable, and decreasing in both of its arguments.

If $V_{t-1}^r(I_{t-1}^a, I_{t-1}) - r_d I_{t-1}^a - cI_{t-1}$ is jointly concave and decreasing in I_{t-1}^a and I_{t-1}, $G_t^r(x,y)$ is decreasing in both x and y. Hence, the same argument as in the proof of Lemma 4 shows that, for any realization of $(\epsilon_t^a, \epsilon_t^m)$,

$$\begin{aligned}
&-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t) \\
=&-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)
\end{aligned}$$

is jointly concave in (y_t^a, x_t, d_t, I_t^a). Concavity is preserved under maximization and expectation, so

$$\mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \le y_t^a \le \min\{K_a + D_t, x_t\}}\{-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t)\}\}$$

is jointly concave in (x_t, d_t, I_t^a). Since $R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a))$ is jointly concave in (d_t, I_t^a), and $\theta(x_t - I_t)^-$ is jointly concave in (x_t, I_t),

$$\begin{aligned}
&R(d_t, I_t^a) + r_d(d_t + \gamma(I_t^a)) - \theta(x_t - I_t)^- - \psi x_t \\
&+ \mathbb{E}_{D_t}\{\max_{\min\{D_t, I_t^a\} \le y_t^a \le \{K_a + D_t, x_t\}}\{-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - D_t, x_t - D_t)\}\}
\end{aligned}$$

is jointly concave in (x_t, d_t, I_t^a). Since concavity is preserved under maximization, $V_t^r(I_t^a, I_t)$ is jointly concave in (I_t^a, I_t).

Next, we show that $V_t^r(I_t^a, I_t) - r_d I_t^a - cI_t$ is decreasing in I_t^a and I_t. Since all of terms in

$-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is decreasing in I_t^a,

it is decreasing in I_t^a itself, if the constraints $\min\{I_t^a, D_t\} \le y_t^a \le \min\{K_a + D_t, x_t\}$ is not binding.

If $y_t^a = I_t^a$,

$$\begin{aligned}
&-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a) \\
=&\phi I_t^a + G_t^r(I_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a).
\end{aligned}$$

If $\phi I_t^a + G_t^r(I_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is strictly increasing in I_t^a,

$-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is strictly increasing in y_t^a

in a small right-neighborhood of I_t^a: $[I_t^a, I_t^a + \xi)$, for a small enough $\xi > 0$. Under this condition, $y_t^a = I_t^a$ is not an optimizer. Hence,

$-(r_d + r_w)(y_t^a - I_t^a)^- + \phi y_t^a + G_t^r(y_t^a - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a, x_t - (d_t + \gamma(I_t^a))\epsilon_t^m - \epsilon_t^a)$ is decreasing in I_t^a,

if it is optimal to choose $y_t^a = I_t^a$.

If $y_t^a = D_t$,

$$
\begin{aligned}
&-(r_d+r_w)(y_t^a-I_t^a)^- +\phi y_t^a + G_t^r(y_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\\
=&-(r_d+r_w)((d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a-I_t^a)^- +\phi((d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a)+G_t^r(0,x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)
\end{aligned}
$$

is decreasing I_t^a.

Analogously, if $y_t^a = K_a + D_t$,

$$
\begin{aligned}
&-(r_d+r_w)(y_t^a-I_t^a)^- +\phi y_t^a + G_t^r(y_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\\
=&-(r_d+r_w)(K_a+(d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a-I_t^a)^- +\phi(K_a+(d_t+\gamma(I_t^a))\epsilon_t^m+\epsilon_t^a)\\
&+G_t^r(K_a,x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)
\end{aligned}
$$

is decreasing in I_t^a.

If $y_t^a = x_t$,

$$
\begin{aligned}
&-(r_d+r_w)(y_t^a-I_t^a)^- +\phi y_t^a + G_t^r(y_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\\
=&-(r_d+r_w)(x_t-I_t^a)^- +\phi x_t + G_t^r(x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)
\end{aligned}
$$

is decreasing in I_t^a.

Hence,

$$
\max_{\min\{D_t,I_t^a\}\le y_t^a\le \min\{x_t,D_t+K_a\}}\{-(r_d+r_w)(y_t^a-I_t^a)^- +\phi y_t^a + G_t^r(y_t^a-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a, x_t-(d_t+\gamma(I_t^a))\epsilon_t^m-\epsilon_t^a)\}
$$

is decreasing in I_t^a. Because, $-\theta(x_t-I_t)^-$ is decreasing in I_t and $F^r(I_1^a)\subset F^r(I_2^a)$ for any $I_1^a \ge I_2^a$,

$$
\begin{aligned}
V_t^r(I_t^a,I_t)-r_dI_t^a-cI_t = &\max_{(x_t,d_t)\in F^r(I_t^a)}\{R(d_t,I_t^a)+r_d(d_t+\gamma(I_t^a))-\theta(x_t-I_t)^- -\psi x_t\\
&+\mathbb{E}_{D_t}\{\max_{\min\{D_t,I_t^a\}\le y_t^a\le\min\{x_t,K_a+D_t\}}\{-(r_d+r_w)(y_t^a-I_t^a)^- +\phi y_t^a\\
&+G_t^r(y_t^a-D_t,x_t-D_t)\}\}\}
\end{aligned}
$$

is decreasing in I_t^a and I_t. This concludes the proof of **part (a)**. **Part (b)** follows directly from the concavity of $V_{t-1}^r(\cdot,y)$ for any y and (12), while **part (c)** follows from the same argument as the proof of Theorem 1. *Q.E.D.*

EC.2. Examples of Concave $R(\cdot,\cdot)$ Functions

In this section, we give some concrete examples of jointly concave $R(\cdot,\cdot)$ functions. We characterize the necessary and sufficient conditions under which $R(\cdot,\cdot)$ is jointly concave for some specific forms of $p(\cdot)$ and $\gamma(\cdot)$. We discuss four families of $p(\cdot)$ and $\gamma(\cdot)$: (a) the inverse demand function, $p(\cdot)$, is a power function and *scarcity function*, $\gamma(\cdot)$, is an exponential function; (b) $p(\cdot)$ is a power function and $\gamma(\cdot)$ is a power function; (c) $p(\cdot)$ is an exponential function and $\gamma(\cdot)$ is an exponential function; and (d) $p(\cdot)$ is an exponential function and $\gamma(\cdot)$ is a power function. These four cases are the most commonly used inverse demand functions and *scarcity functions* in the literature (see, e.g., Sapra et al. (2010)). The results in this section show that the necessary and sufficient condition characterized in Lemma 1 can be satisfied by these popular $p(\cdot)$ and $\gamma(\cdot)$ functions under certain conditions, which are presented in model primitives and easy to verify.

EC.2.1. Power Inverse Demand Function and Exponential Scarcity Function

In this subsection, we specify the functional form of $p(\cdot)$ and $\gamma(\cdot)$ as $p(d_t) = p_0 - (d_t)^\zeta$ and $\gamma(I_t^a) = \gamma_0 - \exp(\eta I_t^a)$. Under Assumptions 1 - 2, the parameters satisfy the following: $\zeta \geq 1$, $\eta \geq 0$, $\underline{d} \geq 0$, $p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta > 0$, and $\underline{d} + \gamma_0 - \exp(\eta K_a) \geq 0$. First, we compute the first and second order derivatives of $p(\cdot)$ and $\gamma(\cdot)$.

$$\begin{cases} p'(d_t) = -\zeta(d_t)^{\zeta-1}, \\ p''(d_t) = -\zeta(\zeta-1)(d_t)^{\zeta-2}, \end{cases} \quad \begin{cases} \gamma'(I_t^a) = -\eta \exp(\eta I_t^a), \\ \gamma''(I_t^a) = -\eta^2 \exp(\eta I_t^a). \end{cases} \tag{EC.17}$$

Note that $-\frac{(\gamma'(I_t^a))^2}{\gamma''(I_t^a)} = \exp(\eta I_t^a) \leq \exp(\eta K_a)$. Hence, the necessary condition characterized in Lemma 2(b) for $R(\cdot,\cdot)$ to be jointly concave is satisfied for this family of $p(\cdot)$'s and $\gamma(\cdot)$'s. Next we characterize the necessary and sufficient condition for $R(\cdot,\cdot)$ to be jointly concave for power inverse demand functions and exponential *scarcity functions*.

LEMMA EC.7. *If* $p(d_t) = p_0 - (d_t)^\zeta$, $\gamma(I_t^a) = \gamma_0 - \exp(\eta I_t^a)$, *with* $\zeta \geq 1$, $\eta \geq 0$, $p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta > 0$, *and* $\underline{d} + \gamma_0 - \exp(\eta K_a) \geq 0$. *We have:*

(a) $R(\cdot,\cdot)$ *is jointly concave on its domain if and only if*

$$p_0 \geq b + \alpha(c + r_d) + (\bar{d})^\zeta + \frac{\zeta \bar{d}^\zeta \exp(\eta K_a)}{(\zeta+1)\bar{d} + (\zeta-1)(\gamma_0 - \exp(\eta K_a))}. \tag{EC.18}$$

(b) Suppose that $\zeta > 1$. $R(\cdot,\cdot)$ *is jointly concave on its domain if and only if*

$$\gamma_0 \geq \exp(\eta K_a) + \frac{\zeta \bar{d}^\zeta \exp(\eta K_a)}{(\zeta-1)(p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta)} - \frac{\zeta+1}{\zeta-1}\bar{d}. \tag{EC.19}$$

Proof:

Part (a). Plug (EC.17) into (3), we have that $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$((\zeta+1)d_t + (\zeta-1)(\gamma_0 - \exp(\eta I_t^a)))(p_0 - b - \alpha(c + r_d) - (d_t)^\zeta) \geq \zeta(d_t)^\zeta \exp(\eta I_t^a), \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \leq K_a.$$

Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$p_0 \geq b + \alpha(c + r_d) + (d_t)^\zeta + \frac{\zeta(d_t)^\zeta \exp(\eta I_t^a)}{(\zeta+1)d_t + (\zeta-1)(\gamma_0 - \exp(\eta I_t^a))}, \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \leq K_a. \tag{EC.20}$$

Since the right hand side of (EC.20) is increasing in d_t and I_t^a, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.18) holds.

Part (b). Since $\zeta > 1$, (EC.19) is equivalent to (EC.18). Therefore, if $\zeta > 1$, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.19) holds. *Q.E.D.*

Lemma EC.7 specifies the necessary and sufficient condition characterized in Lemma 1 in the case with power inverse demand functions and exponential *scarcity functions*. In short, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (a) p_0 is sufficiently large, or (b) $\zeta > 1$ and γ_0 is sufficiently large.

EC.2.2. Power Inverse Demand and Scarcity Functions

In this subsection, we specify the functional form of $p(\cdot)$ and $\gamma(\cdot)$ as $p(d_t) = p_0 - (d_t)^\zeta$, $\gamma(I_t^a) = \begin{cases} \gamma_0 - (I_t^a)^\eta, & \text{for } 0 \le I_t^a \le K_a, \\ \gamma_0, & \text{otherwise.} \end{cases}$ Under Assumptions 1 - 2, the parameters satisfy the following: $\zeta \ge 1$, $\eta \ge 2$, $\underline{d} \ge 0$, $p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta > 0$, and $\underline{d} + \gamma_0 - (K_a)^\eta \ge 0$. First, we compute the first and second order derivatives of $p(\cdot)$ and $\gamma(\cdot)$.

$$\begin{cases} p'(d_t) = -\zeta(d_t)^{\zeta-1}, \\ p''(d_t) = -\zeta(\zeta-1)(d_t)^{\zeta-2}; \end{cases} \tag{EC.21}$$

$$\gamma'(I_t^a) = \begin{cases} -\eta(I_t^a)^{\eta-1}, & \text{if } 0 \le I_t^a \le K_a, \\ 0, & \text{otherwise,} \end{cases} \quad \gamma''(I_t^a) = \begin{cases} -\eta(\eta-1)(I_t^a)^{\eta-2}, & \text{if } 0 \le I_t^a \le K_a, \\ 0, & \text{otherwise.} \end{cases} \tag{EC.22}$$

Note that for $0 < I_t^a \le K_a$, $-\frac{(\gamma'(I_t^a))^2}{\gamma''(I_t^a)} = \frac{\eta}{\eta-1}(I_t^a)^\eta \le \frac{\eta}{\eta-1}(K_a)^\eta$. Hence, the necessary condition characterized in Lemma 2(b) for $R(\cdot,\cdot)$ to be jointly concave is satisfied. Next we characterize the necessary and sufficient condition for $R(\cdot,\cdot)$ to be jointly concave for power inverse demand functions and *scarcity functions*.

Lemma EC.8. *If* $p(d_t) = p_0 - (d_t)^\zeta$, $\gamma(I_t^a) = \begin{cases} \gamma_0 - (I_t^a)^\eta, & \textit{for } 0 \le I_t^a \le K_a, \\ \gamma_0, & \textit{otherwise.} \end{cases}$, *with* $\zeta \ge 1$, $\eta \ge 2$, $p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta > 0$, *and* $\underline{d} + \gamma_0 - (K_a)^\eta \ge 0$. *We have:*

(a) $R(\cdot,\cdot)$ *is jointly concave on its domain if and only if*

$$p_0 \ge b + \alpha(c + r_d) + (\bar{d})^\zeta + \frac{\zeta(\bar{d})^\zeta \eta (K_a)^\eta}{(\eta-1)[(\zeta+1)\bar{d} + (\zeta-1)(\gamma_0 - (K_a)^\eta)]}. \tag{EC.23}$$

(b) Suppose that $\zeta > 1$. $R(\cdot,\cdot)$ *is jointly concave on its domain if and only if*

$$\gamma_0 \ge (K_a)^\eta + \frac{\zeta(\bar{d})^\zeta \eta (K_a)^\eta}{(\eta-1)(\zeta-1)(p_0 - b - \alpha(c + r_d) - \bar{d}^\zeta)} - \frac{\zeta+1}{\zeta-1}\bar{d}. \tag{EC.24}$$

Proof:

Part (a). Plug (EC.21) and (EC.22) into (3), we have that $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$(\eta-1)((\zeta+1)d_t + (\zeta-1)(\gamma_0 - (I_t^a)^\eta))(p_0 - b - \alpha(c + r_d) - (d_t)^\zeta) \ge \zeta(d_t)^\zeta \eta (I_t^a)^\eta,$$

for any $d_t \in [\underline{d}, \bar{d}]$, and $I_t^a \le K_a$. Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$p_0 \ge b + \alpha(c + r_d) + (d_t)^\zeta + \frac{\zeta(d_t)^\zeta \eta (I_t^a)^\eta}{(\eta-1)[(\zeta+1)d_t + (\zeta-1)(\gamma_0 - (I_t^a)^\eta)]}, \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \le K_a. \tag{EC.25}$$

Since the right hand side of (EC.25) is increasing in d_t and I_t^a, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.23) holds.

Part (b). Since $\zeta > 1$, (EC.24) is equivalent to (EC.23). Therefore, if $\zeta > 1$, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.24) holds. *Q.E.D.*

Lemma EC.8 specifies the necessary and sufficient condition characterized in Lemma 1 in the case with power inverse demand and *scarcity* functions. As in the case with power inverse demand functions and exponential *scarcity functions*, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (a) p_0 is sufficiently large, or (b) $\zeta > 1$ and γ_0 is sufficiently large.

EC.2.3. Exponential Inverse Demand and Scarcity Functions

In this subsection, we specify the functional form of $p(\cdot)$ and $\gamma(\cdot)$ as $p(d_t) = p_0 - \exp(\zeta d_t)$ and $\gamma(I_t^a) = \gamma_0 - \exp(\eta I_t^a)$. Under Assumptions 1 - 2, the parameters satisfy the following: $\zeta > 0$, $\eta \geq 0$, $p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d}) > 0$, and $\underline{d} + \gamma_0 - \exp(\eta K_a) \geq 0$. First, we compute the first and second order derivatives of $p(\cdot)$ and $\gamma(\cdot)$.

$$\begin{cases} p'(d_t) = -\zeta \exp(\zeta d_t), \\ p''(d_t) = -\zeta^2 \exp(\zeta d_t), \end{cases} \qquad \begin{cases} \gamma'(I_t^a) = -\eta \exp(\eta I_t^a), \\ \gamma''(I_t^a) = -\eta^2 \exp(\eta I_t^a). \end{cases} \tag{EC.26}$$

Note that $-\frac{(\gamma'(I_t^a))^2}{\gamma''(I_t^a)} = \exp(\eta I_t^a) \leq \exp(\eta K_a)$. Hence, the necessary condition characterized in Lemma 2(b) for $R(\cdot,\cdot)$ to be jointly concave is satisfied for this family of $p(\cdot)$'s and $\gamma(\cdot)$'s. Next we characterize the necessary and sufficient condition for $R(\cdot,\cdot)$ to be jointly concave for exponential inverse demand functions and *scarcity functions*.

Lemma EC.9. *If $p(d_t) = p_0 - \exp(\zeta d_t)$, $\gamma(I_t^a) = \gamma_0 - \exp(\eta I_t^a)$, with $\zeta > 0$, $\eta \geq 0$, $p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d}) > 0$, and $\underline{d} + \gamma_0 - \exp(\eta K_a) \geq 0$. We have $R(\cdot,\cdot)$ is jointly concave on its domain if and only if:*

(a)

$$p_0 \geq b + \alpha(c + r_d) + \exp(\zeta \bar{d}) + \frac{\zeta \exp(\zeta \bar{d}) \exp(\eta K_a)}{\zeta \bar{d} + 2 + \zeta(\gamma_0 - \exp(\eta K_a))}. \tag{EC.27}$$

(b)

$$\gamma_0 \geq \exp(\eta K_a) + \frac{\exp(\zeta \bar{d}) \exp(\eta K_a)}{p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d})} - \frac{\zeta \bar{d} + 2}{\zeta}. \tag{EC.28}$$

Proof:

Part (a). Plug (EC.26) into (3), we have that $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$(\zeta d_t + 2 + \zeta(\gamma_0 - \exp(\eta I_t^a)))(p_0 - b - \alpha(c + r_d) - \exp(\eta d_t)) \geq \zeta \exp(\zeta d_t) \exp(\eta I_t^a), \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \leq K_a.$$

Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$p_0 \geq b + \alpha(c + r_d) + \exp(\zeta d_t) + \frac{\zeta \exp(\zeta d_t) \exp(\eta I_t^a)}{\zeta d_t + 2 + \zeta(\gamma_0 - \exp(\eta I_t^a))}, \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \leq K_a. \tag{EC.29}$$

Since the right hand side of (EC.29) is increasing in d_t and I_t^a, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.27) holds.

Part (b). Since $\zeta > 0$, (EC.28) is equivalent to (EC.27). Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.28) holds. *Q.E.D.*

Lemma EC.9 specifies the necessary and sufficient condition characterized in Lemma 1 in the case with exponential inverse demand functions and *scarcity functions*. In short, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (a) p_0 is sufficiently large, or (b) γ_0 is sufficiently large.

EC.2.4. Exponential Inverse Demand Function and Power Scarcity Function

In this subsection, we specify the functional form of $p(\cdot)$ and $\gamma(\cdot)$ as $p(d_t) = p_0 - \exp(\zeta d_t)$, $\gamma(I_t^a) = \begin{cases} \gamma_0 - (I_t^a)^\eta, & \text{for } 0 \le I_t^a \le K_a, \\ \gamma_0, & \text{otherwise.} \end{cases}$ Under Assumptions 1 - 2, the parameters satisfy the following: $\zeta > 0$, $\eta \ge 2$, $p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d}) > 0$, and $\underline{d} + \gamma_0 - (K_a)^\eta \ge 0$. First, we compute the first and second order derivatives of $p(\cdot)$ and $\gamma(\cdot)$.

$$\begin{cases} p'(d_t) = -\zeta \exp(\zeta d_t), \\ p''(d_t) = -\zeta^2 \exp(\zeta d_t), \end{cases} \tag{EC.30}$$

$$\gamma'(I_t^a) = \begin{cases} -\eta (I_t^a)^{\eta-1}, & \text{if } 0 \le I_t^a \le K_a, \\ 0, & \text{otherwise,} \end{cases} \gamma''(I_t^a) = \begin{cases} -\eta(\eta-1)(I_t^a)^{\eta-2}, & \text{if } 0 \le I_t^a \le K_a, \\ 0, & \text{otherwise.} \end{cases} \tag{EC.31}$$

Note that for $0 < I_t^a \le K_a$, $-\frac{(\gamma'(I_t^a))^2}{\gamma''(I_t^a)} = \frac{\eta}{\eta-1}(I_t^a)^\eta \le \frac{\eta}{\eta-1}(K_a)^\eta$. Hence, the necessary condition characterized in Lemma 2(b) for $R(\cdot,\cdot)$ to be jointly concave is satisfied. Next we characterize the necessary and sufficient condition for $R(\cdot,\cdot)$ to be jointly concave for exponential inverse demand functions and power *scarcity functions*.

Lemma EC.10. *If $p(d_t) = p_0 - \exp(\zeta d_t)$, $\gamma(I_t^a) = \begin{cases} \gamma_0 - (I_t^a)^\eta, & \text{for } 0 \le I_t^a \le K_a, \\ \gamma_0, & \text{otherwise,} \end{cases}$ $\zeta > 0$, $\eta \ge 2$, $p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d}) > 0$, and $\underline{d} + \gamma_0 - (K_a)^\eta \ge 0$. We have $R(\cdot,\cdot)$ is jointly concave on its domain if and only if:*

(a)

$$p_0 \ge b + \alpha(c + r_d) + \exp(\zeta \bar{d}) + \frac{\zeta \exp(\zeta \bar{d}) \eta (K_a)^\eta}{(\eta - 1)[\zeta \bar{d} + 2 + \zeta(\gamma_0 - (K_a)^\eta)]}. \tag{EC.32}$$

(b)

$$\gamma_0 \ge (K_a)^\eta + \frac{\exp(\zeta \bar{d}) \eta (K_a)^\eta}{(\eta - 1)(p_0 - b - \alpha(c + r_d) - \exp(\zeta \bar{d}))} - \frac{\zeta \bar{d} + 2}{\zeta}. \tag{EC.33}$$

Proof:

Part (a). Plug (EC.30) and (EC.31) into (3), we have that $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$(\eta - 1)(\zeta d_t + 2 + \zeta(\gamma_0 - (I_t^a)^\eta))(p_0 - b - \alpha(c + r_d) - \exp(\zeta d_t)) \ge \zeta \exp(\zeta d_t) \eta (I_t^a)^\eta,$$

for any $d_t \in [\underline{d}, \bar{d}]$, and $I_t^a \leq K_a$. Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if

$$p_0 \geq b + \alpha(c + r_d) + \exp(\zeta d_t) + \frac{\zeta \exp(\zeta d_t)\eta(I_t^a)^\eta}{(\eta - 1)[\zeta d_t + 2 + \zeta(\gamma_0 - (I_t^a)^\eta)]}, \text{ for any } d_t \in [\underline{d}, \bar{d}], \text{ and } I_t^a \leq K_a. \quad \text{(EC.34)}$$

Since the right hand side of (EC.34) is increasing in d_t and I_t^a, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.32) holds.

Part (b). Since $\eta \geq 2$ and $\zeta > 0$, (EC.33) is equivalent to (EC.32). Therefore, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (EC.33) holds. *Q.E.D.*

Lemma EC.10 specifies the necessary and sufficient condition characterized in Lemma 1 in the case with exponential inverse demand functions and power *scarcity functions*. As in all three cases above, $R(\cdot,\cdot)$ is jointly concave on its domain if and only if (a) p_0 is sufficiently large, or (b) γ_0 is sufficiently large.

Lemmas EC.7 - EC.10 confirm our previous insight delivered by Lemma 3 that when the price elasticity of demand (i.e., $|\frac{\mathrm{d}d_t/d_t}{\mathrm{d}p_t/p_t}|$) is sufficiently high relative to the *inventory elasticity of demand* (i.e., $|\frac{\mathrm{d}\gamma/\gamma}{\mathrm{d}I_t^a/I_t^a}|$), $R(\cdot,\cdot)$ is jointly concave in (d_t, I_t^a) on its domain. Therefore, Assumption 3 can be satisfied for popular families of $p(\cdot)$'s and $\gamma(\cdot)$'s. Finally, we remark that the above method can be easily adapted to characterize the conditions under which $R(\cdot,\cdot)$ is jointly concave with other families of inverse demand and *scarcity* functions.

References

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